

# FREE MONOTONE TRANSPORT

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**ABSTRACT.** By solving a free analog of the Monge-Ampère equation, we prove a non-commutative analog of Brenier's monotone transport theorem: if an  $n$ -tuple of self-adjoint non-commutative random variables  $Z_1, \dots, Z_n$  satisfies a regularity condition (its conjugate variables  $\xi_1, \dots, \xi_n$  should be analytic in  $Z_1, \dots, Z_n$  and  $\xi_j$  should be close to  $Z_j$  in a certain analytic norm), then there exist invertible non-commutative functions  $F_j$  of an  $n$ -tuple of semicircular variables  $S_1, \dots, S_n$ , so that  $Z_j = F_j(S_1, \dots, S_n)$ . Moreover,  $F_j$  can be chosen to be monotone, in the sense that  $F_j = \mathcal{D}_j g$  and  $g$  is a non-commutative function with a positive definite Hessian. In particular, we can deduce that  $C^*(Z_1, \dots, Z_n) \cong C^*(S_1, \dots, S_n)$  and  $W^*(Z_1, \dots, Z_n) \cong L(\mathbb{F}(n))$ . Thus our condition is a useful way to recognize when an  $n$ -tuple of operators generate a free group factor. We obtain as a consequence that the  $q$ -deformed free group factors  $\Gamma_q(\mathbb{R}^n)$  are isomorphic (for sufficiently small  $q$ , with bound depending on  $n$ ) to free group factors. We also partially prove a conjecture of Voiculescu by showing that free Gibbs states which are small perturbations of a semicircle law generate free group factors. Lastly, we show that entrywise monotone transport maps for certain Gibbs measure on matrices are well-approximated by the matricial transport maps given by free monotone transport.

## 1. INTRODUCTION.

**1.1. On a notion of density in free probability.** Let  $X_1, \dots, X_n$  be classical random variables. Thus  $X = (X_1, \dots, X_n)$  can be viewed as a function defined on a measure space  $(\Omega, \omega)$  with values in (say)  $\mathbb{R}^n$ . Of special interest is the law of  $X$ , which is the measure on  $\mathbb{R}^n$  obtained as the push-forward  $\mu = X_*\omega$ . Very often, it is assumed that  $\mu$  is Lebesgue absolutely continuous and the density  $\rho = d\mu/(\prod dx_j)$  then plays a key role. For example, the density is involved in the classical definition of entropy  $(\int \rho \log \rho \prod dx_j)$ , Fisher information  $(\int \left| \frac{\rho'}{\rho} \right|^2 \prod dx_j)$  and so on.

In passing to the non-commutative case, one assumes that  $X_1, \dots, X_n$  are self-adjoint elements of some finite von Neumann algebra  $(M, \tau)$ , where  $\tau : M \rightarrow \mathbb{C}$  is a normal faithful trace. In other words,  $X_j$  are self-adjoint operators on some Hilbert space  $H$  containing a vector  $\xi$  so that  $\tau(T) = \langle T\xi, \xi \rangle$  satisfies  $\tau(P(X)Q(X)) = \tau(Q(X)P(X))$  for any non-commutative polynomials  $P$  and  $Q$  evaluated at  $X = (X_1, \dots, X_n)$ .

The only (partially) satisfactory extension of the notion of joint law of  $X_1, \dots, X_n$  to the non-commutative case uses the moment method. More precisely, one defines the non-commutative law of  $X_1, \dots, X_n$  as the functional  $\mu$  which assigns to a non-commutative monomial  $P$  the value  $\tau(P(X))$ .

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Unfortunately, no satisfactory replacement notion of *density* has been obtained so far with the exception of the case  $n = 1$ . In this case, spectral theory gives a suitable replacement; however, in absence of any commutation between  $X_1, \dots, X_n$  there is no satisfactory theory of “joint spectrum” even if  $X_j$ ’s all act on a finite-dimensional vector space. This causes a number of problems in free probability theory. For example, Voiculescu introduced two definitions of free entropy ( $\chi$  and  $\chi^*$ ), but for neither of them is there a simple formula analogous to the classical case as soon as  $n \neq 1$ .

**1.2. Free Gibbs states and log-concave measures.** One hint that gives hope that a satisfactory replacement for the notion of density can be found lies in the existence of free analogs of strictly log-concave measures. Log-concave measures on  $\mathbb{R}^n$  are probability laws having particularly nice density: it has the form  $\exp(-V(x_1, \dots, x_n))dx_1 \cdots dx_n$ , where  $V$  is a strictly convex function.

It turns out that these laws have free probability analogs: If  $V$  is a “locally strictly convex non-commutative function” (see [GS09] for a precise definition or section 2.9; one example of such a function is  $V(X_1, \dots, X_n) = \frac{1}{2} \sum X_j^2 + \beta W(X_1, \dots, X_n)$  for a fixed polynomial  $W$  and sufficiently small  $\beta$ ), there exists a unique non-commutative law  $\tau_V$  which satisfies the “Schwinger-Dyson” equation [Gui06, GMS06]

$$\tau_V(P\mathcal{D}V) = \tau_V \otimes \tau_V(\text{Tr}(\mathcal{J}P)).$$

This law  $\tau_V$  is called the *free Gibbs law with potential  $V$* . Here  $\mathcal{D}$  and  $\mathcal{J}$  are suitable non-commutative replacements for the gradient and Jacobian, respectively.

Free Gibbs laws associated to such convex non-commutative functions are very well-behaved. It is worth mentioning that they arise as limits of random matrix models associated to probability measures with log-concave density  $\exp(-N\text{Tr}(V(A_1, \dots, A_n)))$  on the spaces of  $N \times N$  self-adjoint matrices. Alternatively, the law  $\tau_V$  can be characterized as the minimizer of relative free entropy

$$\chi_V(\tau_V) = \chi(\tau_V) - \tau_V(V),$$

where  $\chi(\tau_V)$  is Voiculescu’s (microstates) free entropy [Voi94, Voi02b].

Free Gibbs states have many nice properties with respect to free stochastic calculus, various free differential operators and so on (see e.g. [GMS06, GMS07, GS09]). Thus the fact that, without knowing what a non-commutative density is, we are able to single out a class of non-commutative laws which are similar in property to classical log-concave measures (which by definition are measures with a nice density!) strongly suggests that a non-commutative version of density could be found. Moreover, if it were to be found, such log-concave measures are natural first candidates for a detailed study.

**1.3. Monotone transportation maps as replacements for densities.** There is an alternative way of talking about densities in the classical case, which is a consequence of the following theorem of Brenier: assume that  $\mu$  is a measure on  $\mathbb{R}^n$  satisfying some technical conditions (Lebesgue absolutely continuous, finite second moment, etc.) Let  $\nu$  denote the standard Gaussian measure on  $\mathbb{R}^n$ . Then *there exists a canonical monotone transport map  $\phi_\mu : \mathbb{R}^n \rightarrow \mathbb{R}^n$  from  $\nu$  to  $\mu$* . Here by a *transport map from  $\nu$  to  $\mu$*  we mean a map  $\phi$  satisfying  $\phi_*\nu = \mu$ ; and  $\phi$  is called *monotone* if  $\phi = \nabla\psi$  for some convex function  $\psi$  (roughly speaking, this means that the Jacobian of  $\phi$  is positive-definite almost everywhere). Our point is that such a transport map carries all the information contained in the knowledge of the density  $d\mu/\prod dx_j$  (indeed, this density is essentially the Jacobian of the map). A sign of the fact

that  $\phi_\mu$  is a good analytic object is that it continues to exist even when the density does not (for example, when  $\mu$  is not Lebesgue absolutely continuous, but does not give mass to “small sets”, cf. [Bre91]).

It is thus very tempting to ask if there is any analog of Brenier’s results in the case of non-commutative probability. Here one immediately faces a major obstacle: unlike in the classical case, there are many more non-commutative probability spaces than classical probability spaces. Indeed, up to isomorphism, the unit interval  $[0, 1]$  with Lebesgue measure is the unique non-atomic probability space. At the same time, there are many non-isomorphic non-commutative von Neumann algebras. In fact, there is not even a (separable) von Neumann algebra that contains all others [Oza04], thus there cannot exist a “master law” so that all laws can be obtained as “push-forwards” of that law (as is the case classically). However, as we show in the present paper, one may hope to obtain such a result in certain cases (and, indeed, the Brenier map only exists under certain analytic assumptions about the target measure, even in the classical case).

**1.4. The main results.** In this paper, we give the first examples of existence of non-commutative monotone transport. We thus take the first steps in the study of this subject.

To state our results, recall that the Gaussian law  $\nu$  can be characterized by the following integration by parts formula: for any  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , denoting by  $x$  the vector  $(x_1, \dots, x_n)$ ,

$$\int x \cdot f(x) d\nu(x) = \int \text{Tr}(Jf(x)) d\nu(x),$$

where  $Jf$  denotes the Jacobian of  $f$ . In other words,  $x = J^*I$ , where  $J^*$  denotes the adjoint of  $J$  viewed as an unbounded operator  $L^2(\mathbb{R}^n, \nu) \rightarrow L^2(M_{n \times n}, \nu)$ , and  $I$  denotes the  $n \times n$  identity matrix.

Recall that a free semicircular family  $S = (S_1, \dots, S_n)$  is a free-probability analog of an iid  $n$ -tuple of Gaussian random variables. The semicircle law can also be characterized by the formula of the form

$$S = \mathcal{J}^*(I),$$

where  $\mathcal{J}^*$  is the adjoint of the map  $\mathcal{J}$ , which is a suitable non-commutative replacement for the Jacobian.

Our main theorem shows that monotone transport exists in the non-commutative case, provided that the law of an  $n$ -tuple  $Z = (Z_1, \dots, Z_n)$  is “close” to the semicircle law, in the sense that it “almost satisfies” the equation  $Z = \mathcal{J}^*(I)$ , i.e.,  $Z$  is close to  $\mathcal{J}^*(I)$  in a certain analytic function norm  $\|\cdot\|_A$  (see §2.1). The conditions on  $Z$  automatically imply that the law of  $Z$  is a free “log-concave” Gibbs law.

**Theorem 1.1** (Existence of monotone transport). *Let  $S = (S_1, \dots, S_n)$  be a free semicircular family. If  $Z_1, \dots, Z_n \in (M, \tau)$  are non-commutative random variables such that there exists  $F = F^*$  in the closure of  $\mathcal{A}$  for the norm  $\|\cdot\|_A$  so that  $\mathcal{J}^*(I) = \mathcal{D}F$  is so that  $\|Z - \mathcal{J}^*(I)\|_A$  is sufficiently small, then there exist  $Y_1, \dots, Y_n \in W^*(S_1, \dots, S_n)$  so that:*

- *the law of  $Y = (Y_1, \dots, Y_n)$  is the same as the law of  $Z = (Z_1, \dots, Z_n)$ , so that  $W^*(Z) \cong W^*(Y)$ ;*
- *$W^*(S) = W^*(Y)$  ( $\cong W^*(Z)$ ), so that  $Y_j$  can be viewed as “non-commutative measurable functions” of  $S_1, \dots, S_n$  (we in fact show that these functions can be taken to be non-commutative power series);*
- *$Y_j = \mathcal{D}_j G$  for some  $G \in W^*(S_1, \dots, S_n)$  and  $\mathcal{J}Y$  is positive-definite.*

Here  $\mathcal{D}$  and  $\mathcal{J}$  are suitable non-commutative replacements for the gradient and Jacobian of a function; these differential operators play key roles in free probability theory [Voi98, Voi02b, Voi02a, Gui09, Gui06, AGZ10]). In particular, the last condition is a kind of convexity requirement on  $G$  (since  $\mathcal{J}Y = \mathcal{J}\mathcal{D}G$  is a kind of Hessian of  $G$ ).

By analogy with Brenier's theorem, we call  $F = \mathcal{D}G$  "free monotone transport" from the semicircle law to the law of  $Z$ . We can view the function  $F$  as playing the role of density; the fact that we are able to construct  $F$  in the space of certain analytic maps is a reflection of the fact that the law of  $Z_1, \dots, Z_n$  has a "nice density" relative to the free semicircle law.

By Voiculescu's change of variables formula, one can claim that for laws  $\mu$  satisfying the assumptions of our theorem,

$$\chi(\mu) = \tau \otimes \tau \otimes \text{Tr}(\log \mathcal{J}F_\mu) + \text{universal constant}.$$

We thus see that free entropy can be expressed by a simple formula in terms of the monotone transport map, and is concave in this map.

Our theorem yields a number of isomorphism results, since it gives a rather general condition of when an  $n$ -tuple of operators generates a free group factor. In particular, the theorem applies to free log-concave Gibbs laws with potential of the form  $\frac{1}{2} \sum X_j^2 + \beta W(X_1, \dots, X_n)$  if  $\beta$  is sufficiently small:

**Theorem 1.2.** *Let  $W$  be a fixed self-adjoint polynomial in  $n$  variables and set  $V_\beta = \frac{1}{2} \sum_{j=1}^n X_j^2 + \beta W$ . Let  $\tau_\beta$  be the free Gibbs state with potential  $V_\beta$ . Then for sufficiently small  $\beta$ ,  $W^*(\tau_\beta) \cong L(\mathbb{F}_n)$ .*

This theorem partially answers in the affirmative a conjecture of Voiculescu [Voi06, p. 240] (the full conjecture involves arbitrary values of  $\beta$ ).

We also prove:

**Theorem 1.3.** *Let  $\Gamma_q(\mathbb{R}^n)$  be the von Neumann algebra generated by  $n$   $q$ -deformed semicircular elements of Bozejko and Speicher. Then there are numbers  $q_0(n) > 0$  so that for all  $|q| < q_0(n)$ ,  $\Gamma_q(\mathbb{R}^n) \cong L(\mathbb{F}_n)$ .*

This settles (for small values of  $q$ ) the 20-year old question of the isomorphism class of such  $q$ -deformed von Neumann algebras. Furthermore, since  $L(\mathbb{F}_n)$  is a factor, is non-hyperfinite, strongly solid [OP10] (thus is solid and has no Cartan subalgebras), has the Haagerup property and the complete metric approximation property, (again, for the small values of  $q$ ) our result can be viewed as the culmination of a number of results about these algebras, including [BS91, BKS97, Nou04, Śni01, Śni04, Shl04, Ric05, Shl09, KN11, Dab10, Avs11].

It would be very interesting to extend our main theorem (which is limited by its hypothesis to laws "close" to the semicircle law), and to study the analytical properties of the map  $F_\mu$  (these would be non-commutative analogs of properties of the density associated to a classical law  $\mu$ ). For example,  $F_\mu$  could be used to at least formulate multi-variables analogs of various regularity results in single-variable free probability. Similarly,  $F_\mu$  could play a prominent role in the study of multi-variable random matrix ensembles.

**1.5. Free monotone transport as a limit of classical monotone transport.** As we mentioned before, the free Gibbs law  $\tau_V$  can be obtained as a limit of classical Gibbs measures

in the following sense: for any non-commutative polynomial  $P$ ,

$$\tau_V(P) = \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_V^{(N)}} \left[ \frac{1}{N} \text{Tr}(P(A_1, \dots, A_n)) \right]$$

where  $\mu_V^{(N)}$  is a measure on  $n$ -tuples of self-adjoint  $N \times N$  matrices given by

$$d\mu_V^{(N)} = \exp(-N \text{Tr} V(A_1, \dots, A_n)) \times \text{Lebesgue measure}.$$

Let  $W$  be a fixed self-adjoint non-commutative polynomial and set  $V = \frac{1}{2} \sum X_j^2 + W$ . If we denote by  $\mu^{(N)}$  the Gaussian measure, there exists a unique monotone transport map  $F^{(N)}$  which pushes forward  $\mu^{(N)}$  into  $\mu_V^{(N)}$ . This map operates on matrices entrywise. On the other hand, let  $F$  be the free monotone transport map taking the semicircle law to the free Gibbs state  $\tau_V$ . Using functional calculus,  $F$  induces for each  $N$  a map of  $n$ -tuples of  $N \times N$  matrices (a kind of “matricial” transport). In Theorem 4.7, we show that  $F^{(N)} - F \rightarrow 0$  in  $L^2$ . In other words, the “entrywise” transport is well-approximated by “matricial” transport given by functional calculus on matrices. This is somewhat parallel to Biane’s result [Bia97] which shows that in the Gaussian case, the “entrywise” Segal–Bargmann transform is asymptotically “matricial” and yields the limiting transform in free probability theory.

**1.6. The one-dimensional case.** In the case that  $n = 1$ , so that we are dealing with a single random variable, non-commutativity goes away. In this section, we explain the proof of our main result in the case  $n = 1$ ; indeed, many of the formulae that hold in the general multi-variable case have simple ad hoc explanations in the one-variable case. The arguments in this section are subsumed by the arguments for a general  $n$ ; however, we feel that their inclusion is justified as they serve to clarify our proof of the general case.

We thus start with a semicircular variable  $X$  viewed as the operator of multiplication by  $x$  on  $L^2([-2, 2], \eta)$ , where  $\eta$  is the semicircle law:  $d\eta(x) = \chi_{[-2, 2]}(x) \cdot \frac{1}{2\pi} \sqrt{4 - x^2} dx$ . We write  $\tau(h) = \int h(x) d\eta(x)$ . Thus  $\eta$  is the unique maximizer of the functional

$$(1.1) \quad \chi(\mu) = \iint \log |s - t| d\mu(s) d\mu(t) - \int \frac{1}{2} t^2 d\mu(t).$$

We fix a function  $W$  which is analytic on the disk of radius  $A$  and let  $V(x) = \frac{1}{2} x^2 + W(x)$ . We now consider measure  $\eta_V$  which is the unique maximizer of

$$\chi_V(\mu) = \iint \log |s - t| d\mu(s) d\mu(t) - \int V(t) d\mu(t).$$

If we write  $\tau_V$  for the linear functional  $\tau_V(h) = \int h(x) d\eta_V(x)$ , then  $\tau_V$  is the free Gibbs state associated to  $V$ .

Our goal is to construct a function  $F$  which is analytic on a disk of radius  $A'$  for some  $4 < A' < A$ , so that  $Y = F(X)$  has law  $\eta_V$  when  $X$  has law  $\eta$ . In other words,

$$(1.2) \quad \int h(F(x)) d\eta(x) = \int h(y) d\eta_V(y),$$

or, equivalently,  $F_{\#} \eta = \eta_V$ .

By definition  $\chi_V((x + \delta f)_{\#} \eta_V) \leq \chi_V(\eta_V)$  for all real numbers  $\delta$  which implies that the unique minimizer  $\eta_V$  satisfies the Schwinger-Dyson equation

$$\int V'(t) f(t) d\eta_V(t) = \iint \frac{f(s) - f(t)}{s - t} d\eta_V(s) d\eta_V(t).$$

If we write  $\mathcal{J}f(s, t) = \frac{f(s)-f(t)}{s-t}$ ,  $\mathcal{D}V(x) = V'(x)$  and identify  $L^\infty([-2, 2]^2, d\eta_V(s)d\eta_V(t))$  with  $L^\infty([-2, 2], d\eta_V) \bar{\otimes} L^\infty([-2, 2], d\eta_V)$ , then the Schwinger-Dyson equation reads

$$(1.3) \quad \tau_V(f\mathcal{D}V) = \tau_V \otimes \tau_V(\mathcal{J}f).$$

In the case that  $V$  is strictly convex on a sufficiently large interval (e.g., if  $W$  is sufficiently small), the Schwinger-Dyson equation determines  $\tau_V$  (or, equivalently,  $\eta_V$ ) uniquely [GS09].

Note that (1.3) is equivalent to saying that  $\mathcal{J}^*(1 \otimes 1) = \mathcal{D}V$ . It is not hard to see that  $\mathcal{J}^*(1 \otimes 1) = 2 \int \frac{1}{x-y} d\eta_V(y)$  (here and for the remainder of the section, the improper integrals are taken in the sense of principal value). Thus (1.2) is equivalent to

$$2 \int \frac{1}{x-y} d\eta_V(y) = V'(x) \quad \forall x \in \text{supp}(\eta_V).$$

Replacing  $x$  by  $F(x)$  and  $y$  by  $F(y)$  and remembering that  $F_*\eta = \eta_V$  gives us the following equation for  $F$ :

$$2 \int \frac{1}{F(x) - F(y)} d\eta(y) = V'(F(x)) = F'(x) + W'(F(x)), \quad \forall x \in [-2, 2].$$

To solve this equation, we try to find a solution of the form  $F(x) = x + f(x)$  with  $f$  small if  $W$  is. Then the equation becomes

$$(1.4) \quad 2 \int \frac{1}{x-y} \frac{1}{1 + \mathcal{J}f(x, y)} d\eta(y) = x + f(x) + W'(x + f(x)),$$

where, as before, we use the notation  $\mathcal{J}f(x, y) = \frac{f(x)-f(y)}{x-y}$ .

Using the fact that  $\eta$  is the semicircle law, so that  $2 \int \frac{1}{x-y} d\eta(x) = x$ , we find that (1.4) is equivalent to

$$-2 \int \frac{1}{x-y} \frac{\mathcal{J}f(x, y)}{1 + \mathcal{J}f(x, y)} d\eta(y) = f(x) + W'(x + f(x)).$$

To deal with the difference quotient  $\mathcal{J}f(x, y) = \frac{f(x)-f(y)}{x-y}$ , we add and subtract the term  $f'(x)$ :

$$\begin{aligned} & 2 \int \frac{1}{x-y} (f'(x) - \mathcal{J}f(x, y)) \frac{1}{1 + \mathcal{J}f(x, y)} d\eta(y) \\ &= f'(x) \left[ 2 \int \frac{1}{x-y} \frac{1}{1 + \mathcal{J}f(x, y)} d\eta(x) \right] + f(x) + W'(x + f(x)). \end{aligned}$$

We can substitute (1.4) into the right hand side, to get

$$\begin{aligned} (1.5) \quad & 2 \int \frac{1}{x-y} (f'(x) - \mathcal{J}f(x, y)) \frac{1}{1 + \mathcal{J}f(x, y)} d\eta(y) \\ &= f'(x) [x + f(x) + W'(x + f(x))] + f(x) + W'(x + f(x)) \\ &= (xf)'(x) + \frac{1}{2}(f(x)^2)' + [W(x + f(x))]' . \end{aligned}$$

The resulting equation is equivalent to the previous one under the assumption that  $f'(x) \neq -1$  almost surely.

Since the right-hand side is a full derivative, it is natural to try to express the left-hand side as a full derivative, too. Simple algebra shows that

$$\int \frac{1}{x-y} (f'(x) - \mathcal{J} f(x, y)) \frac{1}{1 + \mathcal{J} f(x, y)} d\eta(y) = \partial_x \int \log(1 + \mathcal{J} f(x, y)) d\eta(y).$$

Substituting this into (1.5) and removing derivatives from both sides finally gives us

$$2 \int \log(1 + \mathcal{J} f(x, y)) d\eta(y) = (xf)(x) + \frac{1}{2}(f(x))^2 + W(x + f(x)) + \text{const.}$$

Finally, we seek  $g$  so that  $f = g'$  solves the previous equation. This gives us the following equation for  $g$ :

$$(1.6) \quad xg'(x) = -W(x + g'(x)) - \frac{1}{2}(g'(x))^2 + 2 \int \log \left( 1 + \frac{g'(x) - g'(y)}{x - y} \right) d\eta(y) + \text{const.}$$

The constant can be fixed by requiring that both sides of the equation vanish at  $x = 0$ . Note that the operator  $\mathcal{N}g = xg'$  multiplies monomials of degree  $n$  by  $n$ . Let  $\Sigma$  be the inverse of  $\mathcal{N}$  (defined on polynomials with zero constant term),  $\Sigma(\hat{g})(x) = \int_0^x y^{-1} \hat{g}(y) dy$ , and let  $\hat{g} = \mathcal{N}g$  so that  $g = \Sigma\hat{g}$ . With this notation, our equation becomes

$$\hat{g}(x) = -W(x + (\Sigma\hat{g})'(x)) - \frac{1}{2}((\Sigma\hat{g})(x))^2 + 2 \int \log \left( 1 + \frac{(\Sigma\hat{g})'(x) - (\Sigma\hat{g})'(y)}{x - y} \right) d\eta(y) + \text{const.}$$

We can now rewrite this equation as a fixed point equation

$$\hat{g}(x) = \Psi_x(\hat{g}(x)).$$

Note that  $\Psi_x(0) = W(x)$ . For  $W$  small, this equation can be solved by iteration in the space of analytic functions converging on a large enough disk. The essential fact here is that  $\hat{g} \mapsto \Psi_x(\hat{g})$  is Lipschitz for a certain analytic norm for any  $x$ . This gives us a function  $\hat{g}$ , which is sufficiently small. Retracing our steps (and noting that for sufficiently small  $W$ ,  $f$  is small, so that  $|f'(x)| < 1$  almost surely), we get back the desired map  $F$  given by  $F = x + f = G'$  if  $G = \frac{1}{2}x^2 + g(x)$ .

It is rather straightforward that if we consider  $V_\beta = \frac{1}{2}x^2 + W_\beta(x)$  for some family  $W_\beta$  analytically depending on  $\beta$  and sufficiently small, then our transport map  $F = F_{V_\beta}$  also depends on  $\beta$  analytically.

It is worth noting that for  $W$  sufficiently small, the map  $F(x) = x + f(x)$  has a positive derivative and is thus monotone. It is therefore the unique monotone map satisfying  $F_*\eta = \eta_V$ . In particular, the map  $F$  gives optimal transport between  $\eta$  and  $\eta_V$ .

It was pointed out to us by Y. Dabrowski that equation (1.6) can be regarded as a free analog of the classical Monge-Ampère equation. To see this, put  $V(x) = \frac{1}{2}x^2$  and  $\tilde{V} = V + W$ , and consider the classical Monge-Ampère equation for the monotone transport map  $H$  from the Gibbs measure  $d\mu = \exp(-V(x))dx$  to  $d\tilde{\mu} = \exp(-\tilde{V}(x))dx$ . Let  $JH$  stand for the Jacobian (derivative) of  $H$ . Then the Monge-Ampère equation reads:

$$\det(JH(x)) = \frac{\exp(-V(x))}{\exp(-\tilde{V}(H(x)))} = \exp(\tilde{V}(H(x)) - V(x)).$$

(We write  $\det JH$  even though  $JH$  is a  $1 \times 1$  matrix to make the analogy clearer). Taking logarithm of both sides and using that  $\log \det A = \text{Tr}(\log A)$  if  $A \geq 0$  gives

$$\text{Tr}(\log JH) = \tilde{V}(H(x)) - V(x).$$

This is quite reminiscent of (1.6), if we rewrite it by setting  $f = g'$ ,  $F(x) = x + f(x)$  and  $\mathcal{J}F = \frac{F(x)-F(y)}{x-y} = 1 + \mathcal{J}f$ :

$$2 \int \log(\mathcal{J}F) d\eta(y) = \left\{ \frac{1}{2}(F(x))^2 + W(F(x)) \right\} - \frac{1}{2}x^2 = \tilde{V}(F(x)) - V(x).$$

Later in the paper, we also consider the  $n$ -variable version of (1.6), which has the form

$$(1 \otimes \tau + \tau \otimes 1) \text{Tr} \log \mathcal{J}F = \mathcal{S} \left[ \left\{ \frac{1}{2} \sum F(X)_j^2 + W(F(X)) \right\} - \frac{1}{2} \sum X_j^2 \right]$$

where  $\mathcal{S}$  is a certain symmetrization operator.

**1.7. Organization of paper.** To prove our theorem, we consider in section §2 a certain space of analytic functions in several non-commuting variables. This is the space in which our construction of the monotone transport map takes place. We also discuss the various differential operators which are the non-commutative replacements for gradients and Jacobians. We finish the section with a kind of implicit function theorem for non-commutative analytic functions.

The next section is devoted to the construction of a non-commutative monotone transport map from the semicircle law  $\tau$  to the law  $\tau_V$  satisfying the Schwinger-Dyson equation with  $V = \frac{1}{2} \sum X_j^2 + W$ . We assume that  $W$  is a given non-commutative analytic function (which we assume to be sufficiently small in a certain norm).

The next section, §4, is devoted to applications. Finally, the last section collects some open questions and problems.

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## 2. NOTATION AND AN IMPLICIT FUNCTION THEOREM.

**2.1. Non-commutative polynomials, power series and norms  $\|\cdot\|_A$ .** We will denote by  $\mathcal{A} = \mathbb{C}\langle X_1, \dots, X_n \rangle$  the algebra of non-commutative polynomials in  $n$  variables. Let  $\mathcal{A}_0 \subset \mathcal{A}$  be the linear span of polynomials with zero constant term.

Following [GMS07], we consider on this space the family of norms  $\|\cdot\|_A$ ,  $A > 1$ , defined as follows. For a monomial  $q$  and arbitrary  $P \in \mathbb{C}\langle X_1, \dots, X_n \rangle$ , let  $\lambda_q(P)$  be the coefficient of  $q$  in the decomposition of  $P$  as monomials; thus  $P = \sum_q \lambda_q(P)q$ . Then we set

$$\|P\|_A = \sum_{q: \deg q \geq 0} |\lambda_q(P)| A^{\deg q}.$$

We'll denote by  $\mathcal{A}^{(A)}$  (resp.,  $\mathcal{A}_0^{(A)}$ ) the completion of  $\mathcal{A}$  (resp.,  $\mathcal{A}_0$ ) with respect to the norm  $\|\cdot\|_A$ . This is a Banach algebra, and can be viewed as the algebra of absolutely convergent power series with radius of convergence at least  $A$ .

Note that the norm  $\|\cdot\|_A$  has the following property: whenever  $T_1, \dots, T_n$  are elements of some Banach algebra  $Q$  and  $\|T_j\|_Q \leq A$ , there exists a contractive map  $\mathcal{A}^{(A)} \rightarrow Q$



sending  $X_j$  to  $T_j$ . Hence, in the following we will see variables as elements of the set of polynomials  $\mathcal{A}^{(A)}$ . However,  $\mathcal{A}^{(A)}$  will still also denote sets of maps, namely the completion of polynomials in the norm  $\|\cdot\|_A$ .

$\mathcal{A}^{op}$  is the opposite algebra, with multiplication defined by  $a^{op}.b^{op} = (ba)^{op}$ . It is equipped with the same norm  $\|\cdot\|_A$ .

**2.2. The operators  $\mathcal{N}$ ,  $\Sigma$ ,  $\Pi$ ,  $\mathcal{J}$ ,  $\partial$  and  $\mathcal{D}$ .** Let us denote by  $\mathcal{N}$  the linear operator on  $\mathcal{A} = \mathbb{C}\langle X_1, \dots, X_n \rangle$  that multiplies a degree  $k$  monomial by  $k$ . Let us also denote by

$$\Sigma : \mathcal{A}_0 \rightarrow \mathcal{A}_0$$

the inverse of  $\mathcal{N}$  pre-composed by the projection  $\Pi$  onto  $\mathcal{A}_0$ , given by

$$\Pi : P \mapsto P - P(0, 0, \dots, 0).$$

If  $g \in \mathcal{A}$  we write  $\mathcal{D}_j g$  for the  $j$ -th cyclic derivative of  $g$  [Voi02a]. For a monomial  $q$ ,  $\mathcal{D}_j q = \sum_{q=AX_jB} BA$ .

Note that  $\mathcal{D}_j g = \mathcal{D}_j \Pi g$ . We'll denote by  $\mathcal{D}g$  the cyclic gradient (viewed as a vector):

$$\mathcal{D}g = (\mathcal{D}_j g)_{j=1}^n.$$

If  $f = (f_1, \dots, f_n)$  with  $f_j \in \mathcal{A}$  then we'll write  $\mathcal{J}f \in M_{n \times n}(\mathcal{A} \otimes \mathcal{A}^{op})$  for the matrix given by

$$\mathcal{J}f = (\partial_j f_i)_{i,j=1}^n,$$

where  $\partial_j$  is the  $j$ -th free difference quotient [Voi98] defined as the derivation from  $\mathcal{A}$  to  $\mathcal{A} \otimes \mathcal{A}^{op}$  satisfying  $\partial_j X_i = \delta_{j=i}$ , that is if  $q$  is a monomial  $\partial_j q = \sum_{q=AX_jB} A \otimes B$ .

**2.3. Notation:  $Tr$ ,  $\mathcal{J}^*$  and  $\sharp$ .** We will often assume that  $\tau$  is a positive trace and also that  $1 \otimes 1 \in \text{dom } \partial_j^*$ , when  $\partial_j : L^2(\mathcal{A}, \tau) \rightarrow L^2(\mathcal{A}, \tau) \bar{\otimes} L^2(\mathcal{A}, \tau)$  is viewed as a densely defined operator. Under this assumption,  $\mathcal{A} \otimes \mathcal{A}^{op}$  belongs to the domain of  $\partial_j^*$  [Voi98]. We set for  $q = \sum a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{A}^{op}$  and  $g \in \mathcal{A}$ ,

$$q \sharp g = \sum_i a_i g b_i.$$

For  $q \in M_{n \times n}(\mathcal{A} \otimes \mathcal{A}^{op})$ , and  $g = (g_j)_{j=1}^n \in \mathcal{A}^n$ , we'll write

$$\begin{aligned} Tr(q) &= \sum_i q_{ii} \in \mathcal{A} \otimes \mathcal{A}^{op}, \\ \mathcal{J}^* q &= \left( \sum_i \partial_i^*(q_{ji}) \right)_{j=1}^n \in L^2(\mathcal{A}, \tau)^n, \\ q \sharp g &= \left( \sum_i q_{ji} \sharp g_i \right)_{j=1}^n. \end{aligned}$$

We will also denote by  $\sharp$  multiplication in  $M_{n \times n}(\mathcal{A} \otimes \mathcal{A}^{op})$  and write  $g \sharp h = \sum g_i h_i$  if  $g, h \in \mathcal{A}^n$ .

**2.4. Non-commutative notions of transport.** Let  $(M, \phi)$  be a von Neumann algebra with trace  $\phi$ , and let  $X_1, \dots, X_n \in M$  be self-adjoint elements generating  $M$ ; thus  $M$  can be regarded as a completion of the algebra  $\mathcal{A}$ , and the trace  $\phi$  induces on  $\mathcal{A}$  a trace  $\tau_X$  (called the non-commutative law of  $X_1, \dots, X_n$ ). We write  $M \cong W^*(\tau_X)$ . Let  $Y_1, \dots, Y_n \in N$  be self-adjoint elements generating another von Neumann algebra  $N$  with trace  $\psi$ ; we denote by  $\tau_Y$  the corresponding linear functional on  $\mathcal{A}$  so that  $N \cong W^*(\tau_Y)$ .

**Definition 2.1.** By *transport* from  $\tau_X$  to  $\tau_Y$  we mean an  $n$ -tuple of self-adjoint elements  $\hat{Y}_1, \dots, \hat{Y}_n \in M$  having the same law as  $Y_1, \dots, Y_n$ . We call such transport *monotone* if  $\hat{Y} = (\hat{Y}_j)_j$  belongs to the  $L^2$ -closure of the set

$$\{\mathcal{D}g : g \in \text{Alg}(X_1, \dots, X_n) \text{ and } \mathcal{J} \mathcal{D}g \geq 0\}.$$

(as part of the definition of monotone transport, we are making the assumption that the variables  $X_1, \dots, X_n$  are making the assumption that  $X_1, \dots, X_n$  are algebraically free, and so  $\mathcal{J} \mathcal{D}g$  is well-defined for any  $g$  in the algebra these variables generate).

When  $M$  is abelian, an element of  $M$  is an (essentially bounded) function of  $X_1, \dots, X_n$  (and  $n = 1$ .) So in the case that  $M$  is abelian, our definition reduces to the statement that the random variables  $\hat{Y}_1, \dots, \hat{Y}_n$  are expressed as (bounded measurable) functions  $\hat{Y}_j = f_j(X_1, \dots, X_n)$ . Since the law of  $Y$  is the same as the law of  $\hat{Y}$ , the push-forward of the measure  $\tau_X$  via the function  $f = (f_j)_j$  is exactly the measure  $\tau_Y$ . So our notion of transport coincides with the classical one in the commutative case.

Our definition of monotone transport is analogous to the classical case where one requires that  $\hat{Y}$  belongs to the closure of the space of gradients of convex functions. However, note that a different notion of derivative is being used. Nonetheless, when  $n = 1$  our requirement reduces to asking that  $\hat{Y} = f(X)$  for some  $f : \mathbb{R} \rightarrow \mathbb{R}$  so that  $f = g'$  for some  $g$  and  $\frac{f(x)-f(y)}{x-y} \geq 0$ , i.e.,  $f$  is monotone and  $g$  convex.

Note that if a transport from  $\tau_X$  to  $\tau_Y$  exists, then there exists a trace-preserving embedding  $N = W^*(Y)$  into  $M = W^*(X)$  given by  $Y_j \mapsto \hat{Y}_j$ .

**2.5. The norm  $\|\cdot\|_{A \otimes_\pi A}$ .** Let us denote by  $\|\cdot\|_{A \otimes_\pi A}$  the projective tensor product norm on  $\mathcal{A}^{(A)} \otimes (\mathcal{A}^{(A)})^{op}$ . In other words

$$\left\| \sum a_i \otimes b_i \right\|_{A \otimes_\pi A} = \sup_{PI} \|PI(\sum a_i \otimes b_i)\|$$

where the supremum is taken over all maps  $PI$  valued in a Banach algebra so that  $PI(1 \otimes b)$  and  $PI(a \otimes 1)$  commute and have norm bounded by the norm of  $b$  and  $a$  respectively. In particular, taking  $PI$  to be given by the right and left multiplication respectively, we see that for all  $q \in \mathcal{A} \otimes \mathcal{A}^{op}$  and  $g \in \mathcal{A}$ , we have

$$\|q \sharp g\|_A \leq \|q\|_{A \otimes_\pi A} \|g\|_A.$$

We will also denote by the same symbol the norm on  $M_{n \times n}(\mathcal{A} \otimes \mathcal{A}^{op})$  (or its completion) given by identifying that space with the Banach space of (left) multiplication operators on  $(\mathcal{A} \otimes \mathcal{A}^{op})^n$ .  $\|\cdot\|_{A \otimes_\pi A}$  also extends to  $(\mathcal{A} \otimes \mathcal{A}^{op})^n$  by putting for  $F = (F_1, \dots, F_n) \in \mathcal{A} \otimes \mathcal{A}^{op}$

$$\|F\|_{A \otimes_\pi A} = \max_{1 \leq i \leq n} \|F_i\|_{A \otimes_\pi A}.$$

In consequence of its definition, we see that the maps

$$\begin{aligned} \# : M_{n \times n}(\mathcal{A} \otimes \mathcal{A}^{op}) \times M_{n \times n}(\mathcal{A} \otimes \mathcal{A}^{op}) &\rightarrow M_{n \times n}(\mathcal{A} \otimes \mathcal{A}^{op}) \\ M_{n \times n}(\mathcal{A} \otimes \mathcal{A}^{op}) \times \mathcal{A}^n &\rightarrow \mathcal{A}^n \end{aligned}$$

are contractive for the norm  $\|\cdot\|_{A \otimes \pi A}$  on  $M_{n \times n}(\mathcal{A} \otimes \mathcal{A}^{op})$  and the norm  $\|\cdot\|_A$  on  $\mathcal{A}^n$ .

## 2.6. Cyclically symmetric functions.

**Definition 2.2.** We say that a polynomial  $q$  in  $\mathcal{A}$  is *cyclically symmetric* if for any  $i_1, \dots, i_k$ , the coefficient of the monomial  $X_{i_1} \cdots X_{i_k}$  in  $q$  is the same as the coefficient of the monomial  $X_{i_k} X_{i_1} \cdots X_{i_{k-1}}$ .

Let  $\mathcal{S}$  denote the operator on  $\mathcal{A}_0$  defined on monomials by

$$\mathcal{S}X_{i_1} \cdots X_{i_p} = \frac{1}{p} \sum_{r=1}^p X_{i_{r+1}} \cdots X_{i_p} X_{i_1} \cdots X_{i_r}.$$

Clearly,  $\mathcal{S}$  sends  $\mathcal{A}_0$  to the space of cyclically symmetric operators. Note also that  $\mathcal{S}g = \mathcal{D}g \# X = \sum_i \mathcal{D}_i g X_i$ . Moreover, one clearly has that  $\|\mathcal{S}g\|_A \leq \|g\|_A$ . Finally, note that  $\mathcal{D}(\mathcal{S}g) = \mathcal{D}g$ .

**2.7. Implicit function theorem.** We next establish a kind of implicit function theorem for elements of the space  $(\mathcal{A}^{(A)})^n$ . For later purpose note that if  $F \in \mathcal{A}^{(A)}$ ,  $F$  is infinitely differentiable in  $\mathcal{A}^{(A')}$  for any  $A' < A$ . These results are most likely folklore, but we were not able to find a precise reference. The proofs follow very closely the classical proof of the implicit function theorem using a contraction mapping principle. For  $F = (F_j)_{j=1}^n \in (\mathcal{A}^{(A)})^n$ , we will use the notation  $\|F\|_A = \sup_j \|F_j\|_A$ . We have denoted by  $\mathcal{A}$  the algebra  $\mathbb{C}\langle X_1, \dots, X_n \rangle$  of non-commutative polynomials in  $n$  variables. Let us write  $\mathcal{A} * \mathcal{A}$  for the algebra of non-commutative polynomials in  $2n$  variables,  $\mathbb{C}\langle X_1, \dots, X_n, Y_1, \dots, Y_n \rangle$ . Also, remember that variables and maps are all seen as converging power series.

**Theorem 2.3** (Implicit Function Theorem). *Assume that  $F = (F_j)_{j=1}^n$  where*

$$F_j(X_1, \dots, X_n, Y_1, \dots, Y_n) \in (\mathcal{A} * \mathcal{A})^{(A)}$$

*is a non-commutative power series in  $2n$  variables, and assume that*

$$F(0, \dots, 0, 0, \dots, 0) = 0.$$

*Let  $Q = \mathcal{J}_2 F(0, \dots, 0)$  (here  $\mathcal{J}_2$  refers to differentiation in the variables  $Y_1, \dots, Y_n$ , with  $X_j$ 's held fixed), and assume that  $Q$  is invertible. Then there exists an  $a_0 \in (0, A)$  and a finite constant  $C$  so that for all  $a \in (0, a_0)$ , any  $U \in \mathcal{A}^{(A)}$  satisfying  $\|U\|_A < a$ , there exists a unique  $V \in (\mathcal{A}^{(A)})^n$  satisfying  $\|V\|_A < Ca$  which solves the equation*

$$F(U, V) = 0.$$

*Moreover, there exists a function  $f \in (\mathcal{A}^{(A)})^n$  so that  $V = f(U)$ .*

*Proof.* Put

$$G(U, V) = V - Q^{-1} \# F(U, V).$$

Then, since  $Q$  is invertible,

$$G(U, V) = V \iff F(U, V) = 0.$$

Moreover, for all  $U, V$  since  $F(0, 0) = 0$ ,

$$G(U, V) = R(U) + H(U, V)$$

with  $R(U) = -Q^{-1} \# \mathcal{J}_1 F(0, \dots, 0) \# U$  and  $H(U, V)$  is a non-commutative power series in  $[(\mathcal{A} * \mathcal{A})^{(A')}]^n$  for any  $A' < A$  whose decomposition in monomials has only monomials of degree greater or equal to two. Therefore, for any  $U, U'$  so that  $\|U\|_A, \|U'\|_A \leq a$  and any  $V, V'$  so that  $\|V\|_A \leq b, \|V'\|_A \leq b$ , with  $a, b \leq A' < A$ , we have

$$(2.1) \quad \|H(U, V)\|_A \leq C(A')(a + b)^2$$

$$(2.2) \quad \|H(U, V) - H(U', V')\|_A \leq C(A')(b\|U - U'\|_A + a\|V - V'\|_A)$$

with a bounded constant  $C(A')$  which is an increasing function of  $A' < A$ . We next fix  $U$  so that  $\|U\|_A \leq a$  and construct a solution by putting  $V_{k+1} = G(U, V_k)$  and  $V_0 = 0$ . By construction, we have by (2.1)

$$\|V_{k+1}\|_A \leq \|R(U)\|_A + \|H(U, V_k)\|_A \leq \|Q^{-1} \# \mathcal{J}_1 F(0, \dots, 0)\|_{A \otimes_\pi A} a + C(A')(a + \|V_k\|_A)^2$$

which shows the existence of a constant  $C$  so that  $\|V_k\|_A \leq Ca$  for all  $k \geq 0$  provided  $a < a_0$  for some  $a_0 > 0$ . Now by (2.2), we get for all  $k \geq 1$ ,

$$\|V_{k+1} - V_k\|_A \leq C(A')a\|V_k - V_{k-1}\|_A$$

and hence  $V_k$  is a Cauchy sequence for  $C(A')a < 1$ . Therefore it converges when  $a \leq a_0 \wedge C(A')^{-1}$ . The limit  $V_*$  satisfies  $\|V_*\|_A \leq Ca$  and  $F(U, V_*) = 0$ , which proves the stated existence. Moreover, by induction we see that for all  $k$ ,  $V_k = f_k(U)$  for some map  $f_k \in \mathcal{A}^{(A)}$  and therefore the limit for  $\|\cdot\|_A$  satisfies the same property since  $\mathcal{A}^{(A)}$  is stable under composition. Finally, this solution is unique by (2.2) which guarantees that two solutions  $V_*$  and  $V'_*$  would satisfy

$$\|V_* - V'_*\|_A \leq C(A')a\|V_* - V'_*\|_A$$

implying that  $V_* = V'_*$  when  $C(A')a < 1$ . □

**2.8. Inverses to non-commutative power series.** The following corollary shows that an “absolutely convergent non-commutative power series” has an “absolutely convergent inverse”:

**Corollary 2.4.** *Let  $A > A'$  and consider the equation  $Y = X + \beta f_\beta(X)$  with  $f_\beta \in (\mathcal{A}^{(A)})^n$  and  $\|X\|_A \leq A'$ . Then for sufficiently small  $\beta$ ,  $\exists G \in (\mathcal{A}^{(A)})^n$  so that  $X = G(Y_\beta)$ .*

*Proof.* We may without loss of generality assume that  $f_\beta(0) = 0$  by replacing  $Y$  with  $Y - \beta f_\beta(0)$ .  $U = \beta Y, V = \beta X$  are solution of

$$F(U, V) = U - V - \beta^2 f_\beta(\beta^{-1}V) = 0.$$

Note that  $F(0, 0) = 0$  and  $Q = \mathcal{J}_2 F(0, 0) = -(I + \beta \mathcal{J} f(0))$  is invertible for sufficiently small  $\beta$ . Moreover,  $\|U\|_A \leq \beta A' + \beta^2 \|f\|_A = \beta A''$  is small when  $\beta$  is and since  $f_\beta \in (\mathcal{A}^{(A)})^n$ , we can apply the implicit function Theorem 2.3 to conclude that the solution exists and is unique in  $\|V\|_A \leq C\beta A''$ . □

**2.9.  $(c, M)$ -convexity.** Let  $V \in \mathcal{A}^{(M)}$  be a non-commutative power series. Following [GS09], we say that  $V$  is  $(c, M)$ -convex if for any  $n$ -tuples of self-adjoint operators  $X = (X_1, \dots, X_n)$  and  $Y = (Y_1, \dots, Y_n)$  acting on a Hilbert space  $H$  and satisfying  $\sup_j \{\|X_j\|, \|Y_j\|\} < \infty$ ,  $V(X)$  is self-adjoint and the operator

$$(\mathcal{D}V(X) - \mathcal{D}V(Y)) \cdot (X - Y) - c(X - Y) \cdot (X - Y)$$

is non-negative. Here we use the notation  $X \cdot Y = \frac{1}{2} \sum X_j Y_j + Y_j X_j$ .

### 3. CONSTRUCTION OF THE FREE MONOTONE TRANSPORT MAP.

**3.1. Outline of the proof of the main result.** To prove existence of the transport map, we start with an  $n$ -tuple of free semicircular elements  $X_1, \dots, X_n$ . We then seek to find some elements  $Y_1, \dots, Y_n$  in the von Neumann algebra  $M$  generated by  $X_1, \dots, X_n$  so that  $Y_1, \dots, Y_n$  have law  $\tau_V$ , which is the unique log-concave free Gibbs law satisfying the Schwinger-Dyson equation with potential  $V$ . We assume that  $V$  has the form  $V = \frac{1}{2} \sum X_j^2 + W$ , and that  $W$  is a “non-commutative analytic function” of  $X_1, \dots, X_n$  which is small in  $\|\cdot\|_A$ .

To construct  $Y = (Y_1, \dots, Y_n)$  we seek  $f = (f_1, \dots, f_n)$  so that  $Y = X + f$  (i.e.,  $Y_j = X_j + f_j$ ). Assuming we could find such  $Y$  with  $f_j$  analytic and of small norm, the Schwinger-Dyson equation for  $Y$  becomes an equation on  $f$ , of the form

$$\mathcal{J}^* \left( \frac{1}{1 + \mathcal{J}f} \right) = X + f + (\mathcal{D}W)(X + f).$$

(Here  $\mathcal{J}^*$  is the adjoint to  $\mathcal{J}$ ). Since for the semicircle law,  $\mathcal{J}^*(1) = X$ , this equation further simplifies as

$$\mathcal{J}^* \left( \frac{\mathcal{J}f}{1 + \mathcal{J}f} \right) + f + (\mathcal{D}W)(X + f) = 0.$$

One is tempted to solve this equation by using the contraction mapping principle; however, one faces the obstacle of bounding the map  $f \mapsto \mathcal{J}^*(\mathcal{J}f/(1 + \mathcal{J}f))$ . Fortunately, under the assumption that  $f = \mathcal{D}g$  and further assumptions on invertibility of  $1 + \mathcal{J}f$ , one is able to rewrite the left hand side of the equation as the cyclic gradient of a certain expression in  $g$ . This gives us an equation for  $g$ . It turns out that then one can use a contraction mapping argument to prove the existence of a solution of this equation for  $g$ ; this is the content of Proposition 3.13. We then retrace our steps, going from  $g$  to  $f = \mathcal{D}g$  and then to  $Y = X + f$ , proving the main result of this section, Theorem 3.15.

Fix  $C > 4$  and let  $W = W^* \in \mathcal{A}^{(C)}$  be a cyclically symmetric function belonging to the closure of  $\mathcal{A}$  in the norm  $\|\cdot\|_A$ . Let  $f \in \mathcal{A}^{(A)}$ , and let  $Y_j = X_j + f_j$ , so that

$$Y = X + f.$$

The main result of this section is that under the assumptions that  $W$  is sufficiently small (i.e.,  $\|W\|_A$  and  $\|\mathcal{J}W\|_{A \otimes \pi A}$  are small),  $Y$  satisfies the equation

$$(3.1) \quad \partial_{Y_j}^*(1 \otimes 1) = Y_j + \mathcal{D}_{Y_j}(W(Y_1, \dots, Y_n))$$

or, in short

$$\mathcal{J}_Y^*(1) = Y + (\mathcal{D}W)(Y)$$

where we denote by  $1$  the identity element of  $M_{n \times n}(M \otimes M^{op})$ . This will imply that  $Y$  has law  $\tau_{\frac{1}{2} \sum X_j^2 + W}$ , giving us the desired transport map (see Theorem 3.16).

To avoid confusion, we will use the following convention: all differential operators ( $\mathcal{D}$ ,  $\mathcal{J}$ ,  $\partial$ ) which either have no indices, or have a numeric index, refer to differentiation with respect to  $X_1, \dots, X_n$ . Operators that involve differentiation with respect to  $Y_1, \dots, Y_n$  will be labeled as  $\mathcal{D}_{Y_j}$ ,  $\partial_{Y_j}$ , etc.

### 3.2. A change of variables formula.

**Lemma 3.1.** *Put  $M = W^*(\mathcal{A}, \tau)$ . Assume that  $Y$  is such that  $\mathcal{J}Y = (\partial_{X_j} Y_i)_{ij} \in M_n(M \bar{\otimes} M^{op})$  is bounded and invertible. Assume further that  $1 \otimes 1$  belongs to the domain of  $\partial_j^*$  for all  $j$ .*

(i) *Define*

$$\hat{\partial}_j(Z) = \sum_{i=1}^n \partial_{X_i}(Z) \# ((\mathcal{J}Y)^{-1})_{ij}$$

where  $\#$  denotes the multiplication

$$(a \otimes b) \# (A \otimes B) = Aa \otimes bB.$$

Then  $\hat{\partial}_j = \partial_{Y_j}$  and  $\partial_{Y_i} X_j = ((\mathcal{J}Y)^{-1})_{ij}$ .

(ii)  $\partial_{Y_j}^*(1 \otimes 1) = \sum_{\ell} \partial_{X_{\ell}}^*((\mathcal{J}Y)^{-1})_{\ell j}^*$ .

(iii) *Assume in addition that  $Y_j = \mathcal{D}_j G$  for some  $G$  in the completion of  $\mathcal{A}$  with respect to  $\|\cdot\|_A$ . Assume that  $G = G^*$ . Let  $(a \otimes b)^{\dagger} = b^* \otimes a^*$ . Then  $(\mathcal{J}Y)_{ij}^{\dagger} = (\mathcal{J}Y)_{ij}$  and  $(\mathcal{J}Y)^{-1}$  is self-adjoint in  $M_n(M \bar{\otimes} M^{op})$ .*

*Proof.* Let  $Q = \mathcal{J}Y$ . (i) We verify that  $\hat{\partial}_j Y_k = \sum_i \partial_{X_i} Y_k \# (Q^{-1})_{ij} = \sum_i Q_{ki} \# (Q^{-1})_{ij} = 1_{k=j} 1 \otimes 1$

To see (ii), we compute

$$\begin{aligned} \langle \partial_{Y_j}^*(1 \otimes 1), X_{i_1} \dots X_{i_p} \rangle &= \langle 1 \otimes 1, \partial_{Y_j}(X_{i_1} \dots X_{i_p}) \rangle \\ &= \sum_{\ell} \langle 1 \otimes 1, \partial_{X_{\ell}}(X_{i_1} \dots X_{i_p}) \# (Q^{-1})_{\ell j} \rangle \\ &= \sum_{\ell} \langle (Q^{-1})_{\ell j}^*, \partial_{X_{\ell}}(X_{i_1} \dots X_{i_p}) \rangle \\ &= \langle \sum_{\ell} \partial_{X_{\ell}}^*(Q^{-1})_{\ell j}^*, X_{i_1} \dots X_{i_p} \rangle \end{aligned}$$

To verify (iii), we first claim that if  $Y = \mathcal{D}G$  for some  $G$  (not necessarily self-adjoint), then  $(\mathcal{J}Y)_{ij} = \sigma[(\mathcal{J}Y)_{ji}]$ , where  $\sigma(a \otimes b) = b \otimes a$ . Indeed, let  $G = X_{i_1} \dots X_{i_k}$ ; in this case

$$\partial_i Y_j = \sum_{G=AX_i B X_j C} CA \otimes B + \sum_{G=AX_j B X_i C} B \otimes CA.$$

Reversing the role of  $i$  and  $j$  amounts to switching the two sums, i.e., applying  $\sigma$ .

We next consider  $G$  of the form

$$G = X_{i_1} \dots X_{i_k} + X_{i_k} \dots X_{i_1} = P + P^*,$$

where  $P = X_{i_1} \dots X_{i_k}$ . In that case

$$Y_j = \sum_{P=AX_j B} BA + A^* B^*$$

and (3.4)

$$(\mathcal{J}Y)_{ji} = \partial_i Y_j = \sum_{P=AX_jB} \sum_{BA=RX_iS} R \otimes S + S^* \otimes R^*.$$

Thus, if we denote by  $*$  the involution  $(a \otimes b)^* = a^* \otimes b^*$ ,

$$(3.2) \quad (\mathcal{J}Y)_{ij}^* = \sigma((\mathcal{J}Y)_{ij}) = (\mathcal{J}Y)_{ji}$$

proving that  $\mathcal{J}Y$  is self-adjoint. Moreover we find that

$$(3.3) \quad ((\mathcal{J}Y)_{ij})^\dagger = \sigma((\mathcal{J}Y)_{ij}^*) = \sigma((\mathcal{J}Y)_{ji}) = (\mathcal{J}Y)_{ij}.$$

where the last two equalities come from (3.2). □

A direct consequence of Lemma 3.1, (ii) and (iii), is the following.

**Corollary 3.2.** *Assume that  $g \in \mathcal{A}^{(A)}$ ,  $1 \otimes 1 \in \text{dom } \partial_j^*$  for all  $j$ , and put  $G = \frac{1}{2} \sum X_j^2 + g$ . Let  $Y_j = X_j + f_j$  so that  $Y = \mathcal{D}G$ . Assume that  $1 + \mathcal{J}f$  is invertible. Then Equation (3.1) is equivalent to the equation*

$$(3.4) \quad \mathcal{J}^* \left( \frac{1}{1 + \mathcal{J}f} \right) = X + f + (\mathcal{D}W)(X + f).$$

### 3.3. An equivalent form of Equation (3.4).

**Lemma 3.3.** *Let  $\tau$  be the semicircle law. Assume that the map  $\xi \mapsto (\mathcal{J}f)\#\xi + \xi$  is invertible on  $(\mathcal{A}^{(A)})^n$ , and that  $f = \mathcal{D}g$  for some  $g = g^*$ . Let*

$$K = -\mathcal{J}^* \circ \mathcal{J} - \text{id}.$$

*Then equation (3.4) is equivalent to*

$$(3.5) \quad K(f) = \mathcal{D}(W(X + f)) + \left[ (\mathcal{J}f)\#f + (\mathcal{J}f)\#\mathcal{J}^* \left( \frac{\mathcal{J}f}{1 + \mathcal{J}f} \right) - \mathcal{J}^* \left( \frac{(\mathcal{J}f)^2}{1 + \mathcal{J}f} \right) \right].$$

*Proof.* Using the formula  $\frac{1}{1+x} = 1 - \frac{x}{1+x}$  and the fact that  $\mathcal{J}^*(1 \otimes 1) = X$ , we see that (3.4) is equivalent to

$$\mathcal{J}^* \left( \frac{\mathcal{J}f}{1 + \mathcal{J}f} \right) + f + (\mathcal{D}W)(X + f) = 0.$$

We will now apply  $(1 + \mathcal{J}f)\#$  to both sides of the equation (this map is, by assumption, invertible, so the resulting equation is equivalent to the one in the previous line):

$$\begin{aligned} \mathcal{J}^* \left( \frac{\mathcal{J}f}{1 + \mathcal{J}f} \right) + f + (\mathcal{D}W)(X + f) + (\mathcal{J}f)\#\mathcal{J}^* \left( \frac{\mathcal{J}f}{1 + \mathcal{J}f} \right) \\ + (\mathcal{J}f)\#f + (\mathcal{J}f)\#(\mathcal{D}W)(X + f) = 0. \end{aligned}$$

Using that  $\frac{x}{1+x} = x - \frac{x^2}{1+x}$ , we get:

$$\begin{aligned} \mathcal{J}^*(\mathcal{J}f) - \mathcal{J}^* \left( \frac{(\mathcal{J}f)^2}{1 + \mathcal{J}f} \right) + f + (\mathcal{D}W)(X + f) \\ + (\mathcal{J}f)\#\mathcal{J}^* \left( \frac{\mathcal{J}f}{1 + \mathcal{J}f} \right) + (\mathcal{J}f)\#f + (\mathcal{J}f)\#(\mathcal{D}W)(X + f) = 0. \end{aligned}$$

Thus we have:

$$K(f) = (\mathcal{D}W)(X+f) + (\mathcal{J}f) \# (\mathcal{D}W)(X+f) \\ + \left[ (\mathcal{J}f) \# f + (\mathcal{J}f) \# \mathcal{J}^* \left( \frac{\mathcal{J}f}{1+\mathcal{J}f} \right) - \mathcal{J}^* \left( \frac{(\mathcal{J}f)^2}{1+\mathcal{J}f} \right) \right].$$

We now note that, because of cyclic symmetry and the fact that  $f = \mathcal{D}g$  with  $g = g^*$ , we may use (3.3) to deduce:

$$\mathcal{D}(W(X+f)) = (\mathcal{D}W)(X+f) + (\mathcal{J}f) \# (\mathcal{D}W)(X+f)$$

Hence (3.4) is equivalent to

$$K(f) = \mathcal{D}(W(X+f)) + \left[ (\mathcal{J}f) \# f + (\mathcal{J}f) \# \mathcal{J}^* \left( \frac{\mathcal{J}f}{1+\mathcal{J}f} \right) - \mathcal{J}^* \left( \frac{(\mathcal{J}f)^2}{1+\mathcal{J}f} \right) \right],$$

as claimed.  $\square$

**3.4. Some identities involving  $\mathcal{J}$  and  $\mathcal{D}$ .** It will be easier to give proofs of several identities involving  $\mathcal{J}$  and  $\mathcal{D}$  if we use the following graphical notation. We will denote the monomial  $X_{i_1} \cdots X_{i_p} \in \mathcal{A}$  by

$$\boxed{i_1 \cdots i_p}.$$

We will also write

$$\boxed{j \ i_1 \cdots i_p}$$

for the element of  $\mathcal{A}^n$  all of whose entries are zero, except that the  $j$ -th one, which is equal to  $X_{i_1} \cdots X_{i_p}$ . We will also write

$$\boxed{\begin{matrix} i_1 \cdots i_p \\ j_q \cdots j_1 \end{matrix}}$$

for the element  $X_{i_1} \cdots X_{i_p} \otimes X_{j_1} \cdots X_{j_q} \in \mathcal{A} \otimes \mathcal{A}^{op}$ , and

$$\boxed{j \ \begin{matrix} i_1 \cdots i_p \\ j_q \cdots j_1 \end{matrix} \ k}$$

for the matrix in  $M_{n \times n}(\mathcal{A} \otimes \mathcal{A}^o)$  whose only nonzero entry is in the  $j$ -th row and  $k$ -th column, and is given by  $X_{i_1} \cdots X_{i_p} \otimes X_{j_1} \cdots X_{j_q}$ .

$$\boxed{j \ \begin{matrix} j_q \cdots j_1 \end{matrix} \ k}$$

simply denotes the case where the entry  $(jk)$  is equal to  $1 \otimes X_{j_1} \cdots X_{j_q}$ .

With these notations, the operators  $\mathcal{D}$  and  $\mathcal{J}$  have nice graphical interpretations:

$$\mathcal{D} \boxed{i_1 \cdots i_p} = \sum_q \boxed{i_q \ i_{q+1} \cdots i_p \ i_1 \cdots i_{q-1}} \\ \mathcal{J} \boxed{j \ i_1 \cdots i_p} = \sum_q \boxed{j \ \begin{matrix} i_1 \cdots i_{q-1} \\ i_p \cdots i_{q+1} \end{matrix} \ i_q}.$$



In particular, if  $g = X_{i_1} \dots X_{i_p}$ , then  $\mathcal{J} \mathcal{D}g$  is given by

$$\mathcal{J} \mathcal{D}g = \sum_{q_1, q_2} \boxed{\begin{matrix} i_{q_2} & i_{q_2+1} \dots i_{q_1-1} & i_{q_1} \\ i_{q_2-1} \dots i_{q_1+1} & & \end{matrix}}$$

where the indices  $i_1, \dots, i_p$  are cyclically arranged around the sides of the box.

Let us use the convention

$$\boxed{j \begin{matrix} i_1 \dots i_p \\ j_q \dots j_1 \end{matrix} k} \cdot \boxed{j \begin{matrix} i'_1 \dots i'_p \\ j'_q \dots j'_1 \end{matrix} k'} = \delta_{k=j'} \boxed{j \begin{matrix} i_1 \dots i_p i'_1 \dots i'_p \\ j_q \dots j_1 j'_q \dots j'_1 \end{matrix} k'}.$$

**Lemma 3.4.** *Assume that  $1 \otimes 1$  belongs to the domain of  $\partial_j^*$  for all  $j$ . Let  $g \in \mathcal{A}^{(A)}$  and let  $f = \mathcal{D}g$ . Then for any  $m \geq -1$  we have:*

$$\begin{aligned} \frac{1}{m+2} \mathcal{D} \left[ (1 \otimes \tau + \tau \otimes 1) \left\{ \sum_{i=1}^n [(\mathcal{J} f)^{m+2}]_{ii} \right\} \right] \\ = -\mathcal{J}^* ((\mathcal{J} f)^{m+2}) + \mathcal{J} f \# \mathcal{J}^* ((\mathcal{J} f)^{m+1}). \end{aligned}$$

*Proof.* Let us say that  $g = \sum_p \sum_{i_1, \dots, i_p} \alpha_{i_1, \dots, i_p}^{(p)} X_{i_1} \dots X_{i_p}$  and  $f = \mathcal{D}g$ . Let us write  $\xi_j = \partial_j^*(1 \otimes 1)$ .

Recall [Voi98, Prop. 4.3] that if  $A \otimes B = X_{i_1} \dots X_{i_p} \otimes X_{j_1} \dots X_{j_q} \in \mathcal{A} \otimes \mathcal{A}^{op}$ , then, if we denote by  $m$  the map  $a \otimes b \otimes c \mapsto abc$ , we have:

$$\begin{aligned} \partial_j^*(A \otimes B) &= (A \otimes B) \# \xi_j - m \circ (1 \otimes \tau \otimes 1) \circ (\partial_j \otimes 1)(A \otimes B) \\ &\quad - m \circ (1 \otimes \tau \otimes 1) \circ (1 \otimes \partial_j)(A \otimes B) \\ &= A \xi_j B - \sum_t \delta_{j=i_t} X_{i_1} \dots X_{i_{t-1}} \tau(X_{i_{t+1}} \dots X_{i_p}) X_{j_1} \dots X_{j_q} \\ &\quad - \sum_s \delta_{j=j_s} X_{i_1} \dots X_{i_p} \tau(X_{j_1} \dots X_{j_{s-1}}) X_{j_{s+1}} \dots X_{j_q}. \end{aligned}$$

Note that  $(\mathcal{J} f)^{m+2} \# \xi_j = \mathcal{J} f \# (\mathcal{J} f)^{m+1} \# \xi_j$ . It follows that

$$\begin{aligned} &[-\mathcal{J}^* ((\mathcal{J} f)^{m+2}) + \mathcal{J} f \# \mathcal{J}^* ((\mathcal{J} f)^{m+1})]_k \\ &= \left[ \sum_l \left\{ m \circ (1 \otimes \tau \otimes 1) \circ (\partial_l \otimes 1) [(\mathcal{J} f)^{m+2}]_{kl} \right. \right. \\ &\quad \left. \left. - \sum_r (\mathcal{J} f)_{kr} \# m \circ (1 \otimes \tau \otimes 1) \circ (\partial_l \otimes 1) [(\mathcal{J} f)^{m+1}]_{rl} \right\} \right. \\ &\quad \left. + \left[ \sum_l \left\{ m \circ (1 \otimes \tau \otimes 1) \circ (1 \otimes \partial_l) [(\mathcal{J} f)^{m+2}]_{kl} \right. \right. \right. \\ &\quad \left. \left. - \sum_r (\mathcal{J} f)_{kr} \# m \circ (1 \otimes \tau \otimes 1) \circ (1 \otimes \partial_l) [(\mathcal{J} f)^{m+1}]_{rl} \right\} \right] \\ &= (I) + (II). \end{aligned}$$

We will concentrate on the first term, the other term being similar. It is not hard to see that, by the derivation property  $\partial_l$  and the fact that  $\mathcal{J}f\#(\mathcal{J}f)^{m+1} = (\mathcal{J}f)^{m+2}$ ,

$$\begin{aligned} & m \circ (1 \otimes \tau \otimes 1) \circ (\partial_l \otimes 1) ([(\mathcal{J}f)^{m+2}]_{kl}) \\ & - \sum_r (\mathcal{J}f)_{kr} \# m \circ (1 \otimes \tau \otimes 1) \circ (\partial_l \otimes 1) ([(\mathcal{J}f)^{m+1}]_{rl}) \\ & = \sum_r m \circ (1 \otimes \tau \otimes 1) ([(\partial_l \otimes 1)(\mathcal{J}f)_{kr}] \#_2 [(\mathcal{J}f)^{m+1}]_{rl}) \end{aligned}$$

where we use the notation  $a \otimes b \otimes c \#_2 p \otimes q = a \otimes bp \otimes qc$ . Hence (using the notation  $a \otimes b \# 1 = ab$ ), we find

$$\begin{aligned} (I) = & \sum_{p_1 \dots p_{m+2}} \sum_{i_1^{(1)}, \dots, i_{p_1}^{(1)}} \dots \sum_{i_1^{(m+2)}, \dots, i_{p_{m+2}}^{(m+2)}} \sum_{q_1 \neq r_1=1}^{p_1} \dots \sum_{q_{m+2} \neq r_{m+2}=1}^{p_{m+2}} \delta_{i_{q_1}^{(1)}=k} \prod_{u=1}^{m+2} \alpha_{i_1^{(u)} \dots i_{p_u}^{(u)}}^{(p_u)} \\ & \sum_{q_1 < t < r_1} \boxed{\begin{matrix} i_{q_1+1}^{(1)} \dots i_{t-1}^{(1)} & i_{r_1}^{(1)} \\ i_{q_1-1}^{(1)} \dots i_{r_1+1}^{(1)} \end{matrix}} \cdot \boxed{\begin{matrix} i_{q_2}^{(2)} & & i_{r_2}^{(2)} \\ i_{q_2-1}^{(2)} \dots i_{r_2+1}^{(2)} \end{matrix}} \dots \boxed{\begin{matrix} i_{q_{m+2}}^{(m+2)} & & i_{r_{m+2}+1}^{(m+2)} \\ i_{q_{m+2}-1}^{(m+2)} \dots i_{r_{m+2}+1}^{(m+2)} \end{matrix}} \# 1 \\ & \times \delta_{i_{r_{m+2}}^{(m+2)}=i_t^{(1)}} \tau(X_{i_{t+1}^{(1)}} \dots X_{i_{r_1-1}^{(1)}} \cdot X_{i_{q_2+1}^{(2)}} \dots X_{i_{r_2-1}^{(2)}} \dots X_{i_{q_{m+2}+1}^{(m+2)}} \dots X_{i_{r_{m+2}-1}^{(m+2)}}). \end{aligned}$$

Thus

$$\begin{aligned} (I) = & \sum_{p_1 \dots p_{m+2}} \sum_{i_k^{(j)}, q_j, r_j} \delta_{i_{q_1}^{(1)}=k} \prod_{u=1}^{m+2} \alpha_{i_1^{(u)} \dots i_{p_u}^{(u)}}^{(p_u)} \sum_{q_1 < t < r_1} \boxed{\begin{matrix} i_{q_1-1}^{(1)} \dots i_{r_1+1}^{(1)} & i_{r_1}^{(1)} \end{matrix}} \cdot \\ & \boxed{\begin{matrix} i_{q_2}^{(2)} & & i_{r_2}^{(2)} \\ i_{q_2-1}^{(2)} \dots i_{r_2+1}^{(2)} \end{matrix}} \dots \boxed{\begin{matrix} i_{q_{m+2}}^{(m+2)} & & i_{r_{m+2}+1}^{(m+2)} \\ i_{q_{m+2}-1}^{(m+2)} \dots i_{r_{m+2}+1}^{(m+2)} \end{matrix}} \cdot \boxed{\begin{matrix} i_{t-1}^{(1)} \dots i_{q_1+1}^{(1)} \end{matrix}} \# 1 \\ & \times \delta_{i_{r_{m+2}}^{(m+2)}=i_t^{(1)}} \tau(X_{i_{t+1}^{(1)}} \dots X_{i_{r_1-1}^{(1)}} \dots X_{i_{q_2+1}^{(2)}} \dots X_{i_{r_2-1}^{(2)}} \dots X_{i_{q_{m+2}+1}^{(m+2)}} \dots X_{i_{r_{m+2}-1}^{(m+2)}}). \end{aligned}$$

It follows that

$$\begin{aligned}
(I) &= \sum_{p_1 \dots p_{m+2}} \sum_{i_k^{(j)}, q_j, r_j} \delta_{i_{q_1}^{(1)}=k} \prod_{u=1}^{m+2} \alpha_{i_1^{(u)} \dots i_{p_u}^{(u)}}^{(p_u)} \sum_{q_1 < t < r_1} \boxed{i_{q_1-1}^{(1)} \dots i_{r_1+1}^{(1)} i_{r_1}^{(1)}} \\
&\quad \boxed{i_{q_2}^{(2)} i_{q_2-1}^{(2)} \dots i_{r_2+1}^{(2)} i_{r_2}^{(2)}} \dots \boxed{i_{q_{m+2}}^{(m+2)} i_{q_{m+2}-1}^{(m+2)} \dots i_{r_{m+2}+1}^{(m+2)} i_{r_{m+2}}^{(m+2)}} \boxed{i_t^{(1)} i_{t-1}^{(1)} \dots i_{q_1+1}^{(1)}} \#1 \\
&\quad \times \tau(X_{i_{t+1}^{(1)}} \dots X_{i_{r_1-1}^{(1)}} \dots X_{i_{q_2+1}^{(2)}} \dots X_{i_{r_2-1}^{(2)}} \dots X_{i_{q_{m+2}+1}^{(m+2)}} \dots X_{i_{r_{m+2}-1}^{(m+2)}}) \\
&= \frac{1}{m+2} \sum_{p_1 \dots p_{m+2}} \sum_{i_k^{(j)}} \sum_{t \neq r_1=1}^{p_1} \dots \sum_{q_{m+2} \neq r_{m+2}=1}^{p_{m+2}} \prod_{u=1}^{m+2} \alpha_{i_1^{(u)} \dots i_{p_u}^{(u)}}^{(p_u)} \sum_i \\
&\quad \mathcal{D}_k \left[ \boxed{i_t^{(1)} i_{t-1}^{(1)} \dots i_{r_1+1}^{(1)} i_{r_1}^{(1)}} \cdot \boxed{i_{q_2}^{(2)} i_{q_2-1}^{(2)} \dots i_{r_2+1}^{(2)} i_{r_2}^{(2)}} \dots \boxed{i_{q_{m+2}}^{(m+2)} i_{q_{m+2}-1}^{(m+2)} \dots i_{r_{m+2}+1}^{(m+2)} i_{r_{m+2}}^{(m+2)}} \right]_{ii} \\
&\quad \times \tau(X_{i_{t+1}^{(1)}} \dots X_{i_{r_1-1}^{(1)}} \dots X_{i_{q_2+1}^{(2)}} \dots X_{i_{r_2-1}^{(2)}} \dots X_{i_{q_{m+2}+1}^{(m+2)}} \dots X_{i_{r_{m+2}-1}^{(m+2)}}) \\
&= \frac{1}{m+2} \mathcal{D}_k((1 \otimes \tau)(\text{Tr}[(\mathcal{J}f)^{m+2}])),
\end{aligned}$$

where in the next-to-last equality we note that when the operator  $\mathcal{D}$  is applied, the various “boxes” are cyclically rearranged producing  $m+2$  terms which are identical after permuting the indices in the sums over  $p_1, \dots, p_{m+2}$  and  $q_1, \dots, q_{m+2}$ .  $\square$

**Lemma 3.5.** Assume that  $f = \mathcal{D}g$ ,  $g = g^* \in \mathcal{A}^{(A)}$ , and assume that  $1 \otimes 1 \in \text{dom } \partial_j^*$  for all  $j$ . Assume that  $\|\mathcal{J}f\|_{A \otimes \pi A} < 1$  and let

$$Q(g) = \left[ (1 \otimes \tau + \tau \otimes 1) \left\{ \sum_{i=1}^n (\mathcal{J}f)_{ii} - (\log(1 + \mathcal{J}f))_{ii} \right\} \right].$$

Then

$$\mathcal{D}Q = \mathcal{J}f \# \mathcal{J}^* \left( \frac{\mathcal{J}f}{1 + \mathcal{J}f} \right) - \mathcal{J}^* \left( \frac{(\mathcal{J}f)^2}{1 + \mathcal{J}f} \right).$$

*Proof.* We use Lemma 3.4 and compare the (converging) Taylor series expansions of both sides term by term.  $\square$

**Lemma 3.6.** Assume that  $\tau$  is the semicircle law. Let

$$K(f) = -\mathcal{J}^*(\mathcal{J}f) - f.$$

Assume that  $f = \mathcal{D}g$ , and that  $g \in \mathcal{A}$  is cyclically symmetric. Then

$$\begin{aligned}
K(f) &= \mathcal{D} \left\{ (1 \otimes \tau + \tau \otimes 1) \left( \sum_i [\mathcal{J}f]_{ii} \right) - f \# X \right\} \\
&= \mathcal{D} \{ (1 \otimes \tau + \tau \otimes 1) (\text{Tr}[\mathcal{J}\mathcal{D}g]) - \mathcal{N}g \}.
\end{aligned}$$

*Proof.* When  $m = -1$ , the equality in Lemma 3.4 becomes

$$\mathcal{D} \{ (1 \otimes \tau + \tau \otimes 1) (\text{Tr}[\mathcal{J}f]) \} = -\mathcal{J}^*(\mathcal{J}f) + \mathcal{J}f \# \mathcal{J}^*(I),$$

where  $I \in M_{n \times n}(\mathcal{A} \otimes \mathcal{A}^{op})$  denotes the identity matrix. Since  $\tau$  is the semicircle law,  $\mathcal{J}^*(I) = X$ , and  $\mathcal{J}f \# X = \mathcal{N}f$  (here  $\mathcal{N}$  is applied entrywise to the vector  $f$ ).

On the other hand, since  $\mathcal{D}$  reduces the degree of polynomials by 1,  $\mathcal{N}\mathcal{D}g = \mathcal{D}\mathcal{N}g - \mathcal{D}g = \mathcal{D}\mathcal{N}g - f$ . Thus we deduce

$$\begin{aligned} \mathcal{D}\{(1 \otimes \tau + \tau \otimes 1)(\text{Tr}[\mathcal{J}\mathcal{D}g]) - \mathcal{N}g\} &= -\mathcal{J}^*(\mathcal{J}f) + \mathcal{N}\mathcal{D}g - \mathcal{D}\mathcal{N}g \\ &= K(f). \end{aligned}$$

Using the equality  $f \# X = \mathcal{D}g \# X = \mathcal{N}g$ , we get the statement of the Lemma.  $\square$

**Lemma 3.7.** *Let  $\tau$  be the semicircle law. Assume that  $f = \mathcal{D}g$  for some  $g = g^*$  cyclically symmetric and that  $\|\mathcal{J}f\|_{A \otimes_{\pi} A} < 1$ . Let*

$$Q(g) = [(1 \otimes \tau + \tau \otimes 1)\text{Tr}\{(\mathcal{J}\mathcal{D}g) - (\log(1 + \mathcal{J}\mathcal{D}g))\}].$$

*Then Equation (3.4) is equivalent to the equation*

$$\mathcal{D}\{(1 \otimes \tau + \tau \otimes 1)(\text{Tr}[\mathcal{J}\mathcal{D}g]) - \mathcal{N}g\} = \mathcal{D}(W(X + \mathcal{D}g)) + \mathcal{D}Q(g) + \mathcal{J}\mathcal{D}g \# \mathcal{D}g.$$

*Proof.* By Lemma 3.6, the left-hand side is precisely  $K(f)$ . By Lemma 3.3, we obtain

$$K(f) = \mathcal{D}(W(X + f)) + \left( \mathcal{J}f \# \mathcal{J}^* \left( \frac{\mathcal{J}f}{1 + \mathcal{J}f} \right) - \mathcal{J}^* \left( \frac{(\mathcal{J}f)^2}{1 + \mathcal{J}f} \right) + \mathcal{J}f \# f \right)$$

which proves the statement.  $\square$

We now turn to the proof of existence of a  $g$  that satisfies the equation above. We will use a fixed-point argument; thus we first give some estimates on differential operators that will be involved.

**3.5. Technical estimates on certain differential operators.** For the remainder of the section, we will assume that  $\tau : \mathcal{A} \rightarrow \mathbb{C}$  satisfies

$$(3.6) \quad |\tau(q)| \leq C_0^{\deg q}$$

for any monomial  $q \in \mathcal{A}$ . We begin with a few technical estimates on certain differential operators.

**Lemma 3.8.** *Let  $g_1, \dots, g_m \in \mathcal{A}_0$ . Set*

$$Q_m(g_1, \dots, g_m) = (1 \otimes \tau + \tau \otimes 1) \left\{ \sum_{i=1}^n [(\mathcal{J}\mathcal{D}g_1) \cdots (\mathcal{J}\mathcal{D}g_m)]_{ii} \right\}.$$

*Assume that (3.6) holds and moreover that  $C_0/A < 1/2$ . Then*

$$\|Q_m(\Sigma g_1, \dots, \Sigma g_m)\|_A \leq 2(2A^{-2})^m \prod_{k=1}^m \|g_k\|_A.$$

*In particular,  $Q_m$  extends to a bounded multilinear operator on  $\mathcal{A}^{(A)}$  with values in  $\mathcal{A}^{(A)}$ .*

*Proof.* Note that by definition, assuming that  $g$  is a monomial, we have

$$\begin{aligned} [\mathcal{J}\mathcal{D}\Sigma g]_{ji} &= \frac{1}{\deg g} \partial_i \mathcal{D}_j g \\ &= \frac{1}{\deg g} \sum_{g=AX_jB} \sum_{BA=RX_iQ} R \otimes Q. \end{aligned}$$

Thus if  $g_1, \dots, g_m$  are monomials, then

$$[(\mathcal{J} \mathcal{D} \Sigma g_1) \cdots (\mathcal{J} \mathcal{D} \Sigma g_m)]_{j_0 j_m} = \frac{1}{\prod_t \deg g_t} \sum_{\substack{j_1, \dots, j_{m-1} \\ g_k = A_k X_{j_k} B_k}} \sum_{B_k A_k = R_k X_{j_{k+1}} Q_k} R_1 \cdots R_m \otimes Q_1 \cdots Q_m$$

and so, using the fact that for a given  $j_k$  there can be at most  $\deg g_k$  decompositions of  $g_k$  as  $A_k X_{j_{k+1}} B_k$ , and the degree of  $A_k$  determines such decompositions, we conclude that each such sum has at most  $\prod_t \deg g_t$  nonzero terms, so that

$$\begin{aligned} & \| (1 \otimes \tau) [(\mathcal{J} \mathcal{D} \Sigma g_1) \cdots (\mathcal{J} \mathcal{D} \Sigma g_m)]_{j_0 j_m} \|_A \\ & \leq \frac{1}{\prod_t \deg g_t} \sum_{\substack{j_1, \dots, j_{m-1} \\ g_k = A_k X_{j_k} B_k}} \sum_{B_k A_k = R_k X_{j_{k+1}} Q_k} A^{\deg(R_1 \cdots R_m)} \otimes |\tau(Q_1 \cdots Q_m)| \\ & \leq \sum_{l_1, \dots, l_m} A^{l_1 + \dots + l_m} \cdot C_0^{\deg g_1 - l_1 - 2 + \dots + \deg g_m - l_m - 2} \\ & = A^{\deg g_1 + \dots + \deg g_m} A^{-2m} \sum_{l_1 \cdots l_{m-1}} \left( \frac{C_0}{A} \right)^{\deg g_1 - l_1 - 2 + \dots + \deg g_m - l_m - 2} \\ & \leq \prod_{k=1}^m [A^{-2} \|g_k\|_A \frac{1}{1 - C_0/A}] \leq \prod_{k=1}^m [2A^{-2} \|g_k\|_A]. \end{aligned}$$

A similar estimate holds for  $(\tau \otimes 1) [(\mathcal{J} \mathcal{D} \Sigma g_1) \cdots (\mathcal{J} \mathcal{D} \Sigma g_m)]_{j_0 j_m}$ .

Since  $Q_m(\Sigma g_1, \dots, \Sigma g_m)$  is multi-linear in  $g_1, \dots, g_m$ , we find that if  $g_j = \sum \lambda(j, q) q$  is the decomposition of  $g_j$  in terms of monomials, then we get that

$$Q_m(\Sigma g_1, \dots, \Sigma g_m) = \sum_{q_1, \dots, q_m} \lambda(1, q_1) \cdots \lambda(m, q_m) Q_m(\Sigma q_1, \dots, \Sigma q_m)$$

and so

$$\begin{aligned} \|Q_m(\Sigma g_1, \dots, \Sigma g_m)\|_A & \leq 2(2A^{-2})^m \sum_{q_1, \dots, q_m} \prod_{j=1}^m |\lambda(j, q_j)| \|q_j\|_A \\ & = 2(2A^{-2})^m \prod_{j=1}^m \sum_{q_j} |\lambda(j, q_j)| \|q_j\|_A \\ & = 2(2A^{-2})^m \prod_{j=1}^m \|g_j\|_A. \end{aligned}$$

This concludes the proof for arbitrary  $g_1, \dots, g_m$ .  $\square$

**Lemma 3.9.** *For  $g, f \in \mathcal{A}_0$ , set  $Q_m(\Sigma g) = Q_m(\Sigma g, \dots, \Sigma g)$ . Assume that (3.6) holds and moreover that  $C_0/A < 1/2$ . Then*

$$\|Q_m(\Sigma g) - Q_m(\Sigma f)\|_A \leq 2(2A^{-2})^m \sum_{k=0}^{m-1} \|g\|_A^k \|f\|_A^{m-k-1} \|f - g\|_A$$

In particular,  $\|Q_m(\Sigma g)\|_A \leq 2(2A^{-2})^m \|g\|_A^m$ .

*Proof.* Let  $\kappa = 2(2A^{-2})^m$ . Then

$$\begin{aligned} \|Q_m(\Sigma f) - Q_m(\Sigma g)\|_A &= \left\| \sum_{k=0}^{m-1} Q_m(\underbrace{\Sigma g, \dots, \Sigma g}_k, \underbrace{\Sigma f, \dots, \Sigma f}_{m-k}) - Q_m(\underbrace{\Sigma g, \dots, \Sigma g}_{k+1}, \underbrace{\Sigma f, \dots, \Sigma f}_{m-k-1}) \right\|_A \\ &\leq \sum_{k=0}^{m-1} \|Q_m(\underbrace{\Sigma g, \dots, \Sigma g}_k, \Sigma f - \Sigma g, \underbrace{\Sigma f, \dots, \Sigma f}_{m-k-1})\|_A \\ &\leq \kappa \sum_{k=0}^{m-1} \|g\|_A^k \|f\|_A^{m-k-1} \|f - g\|_A. \end{aligned}$$

□

**Corollary 3.10.** *Assume that (3.6) holds and moreover that  $C_0/A < 1/2$ . Then the maps  $g \mapsto Q_m(\Sigma g)$  extend by continuity to the completion of  $\mathcal{A}_0$  with respect to  $\|\cdot\|_A$ .*

**Lemma 3.11.** *Assume that (3.6) holds and moreover that  $C_0/A < 1/2$ . Let  $g \in \mathcal{A}_0^{(A)}$  be such that  $\|g\|_A < \frac{A^2}{2}$ , and set*

$$Q(\Sigma g) = \sum_{m \geq 0} \frac{(-1)^m}{m+2} Q_{m+2}(\Sigma g).$$

*Then this series converges in  $\|\cdot\|_A$ . Moreover, in the sense of analytic functional calculus on  $M_{n \times n}(W^*(\mathcal{A} \otimes \mathcal{A}^{op}, \tau \otimes \tau^{op}))$ , we have equality*

$$Q(\Sigma g) = \left[ (1 \otimes \tau + \tau \otimes 1) \left\{ \sum_{i=1}^n (\mathcal{J} \mathcal{D} \Sigma g)_{ii} - (\log(1 + \mathcal{J} \mathcal{D} \Sigma g))_{ii} \right\} \right].$$

*Furthermore, the function  $Q$  satisfies the (local) Lipschitz condition on  $\{g : \|g\|_A \leq A^2/2\}$*

$$\|Q(\Sigma g) - Q(\Sigma f)\|_A \leq \|f - g\|_A \frac{2}{A^2} \left( \frac{1}{\left(1 - \frac{2\|f\|_A}{A^2}\right) \left(1 - \frac{2\|g\|_A}{A^2}\right)} - 1 \right),$$

*and the bound*

$$\|Q(\Sigma g)\|_A \leq \frac{\left(\frac{2\|g\|_A}{A^2}\right)^2}{1 - \frac{2\|g\|_A}{A^2}}.$$

*Proof.* Let  $\kappa = \frac{A^2}{2\|g\|_A}$  and  $\lambda = \|g\|_A$ . By Lemma 3.8, we have that  $\|Q_{m+2}(g)\|_A \leq 2(\lambda/\kappa)^{(m+2)}$ . Thus, as we assumed  $\lambda < \kappa$ , the series defining  $Q(\Sigma g)$  converges. To see the claimed equality with the expression given by functional calculus, we only need to note that  $\log(1 +$

$x) = -\sum_{m \geq 1} \frac{(-x)^m}{m}$ . Finally, since  $m+2 \geq 2$  in our series, we obtain that

$$\begin{aligned} \|Q(\Sigma g) - Q(\Sigma f)\|_A &\leq \sum_{m \geq 0} \frac{1}{m+2} \|Q_{m+2}(\Sigma g) - Q_{m+2}(\Sigma f)\|_A \\ &\leq \|f - g\|_A \sum_{m \geq 0} \sum_{k=0}^{m+1} (2A^{-2})^{m+2} \|f\|_A^{m-k+1} \|g\|_A^k \\ &\leq \|f - g\|_A (2A^{-2}) \left( \sum_{l \geq 0} \sum_{k \geq 0} (2A^{-2})^l \|f\|_A^l (2A^{-2})^k \|g\|_A^k - 1 \right), \end{aligned}$$

where we have written  $m = l + k - 1$  which is non-negative precisely when  $l$  and  $k$  are not both zero. Thus, if  $\|f\|_A \leq A^2/2$  and  $\|g\|_A \leq A^2/2$ , we deduce that

$$\|Q(\Sigma g) - Q(\Sigma f)\|_A \leq \|f - g\|_A \frac{2}{A^2} \left( \frac{1}{\left(1 - \frac{2\|f\|_A}{A^2}\right) \left(1 - \frac{2\|g\|_A}{A^2}\right)} - 1 \right).$$

Setting  $f = 0$  gives us the estimate

$$\|Q(\Sigma g)\|_A \leq \frac{2\|g\|_A}{A^2} \left( \frac{1}{1 - \frac{2\|g\|_A}{A^2}} - 1 \right),$$

as claimed.  $\square$

**Corollary 3.12.** *Let  $g \in \mathcal{A}_0^{(A)}$ , assume that  $\|g\|_A < A^2/2$  and let  $B \geq A + \|g\|_A$ . Let  $W \in \mathcal{A}^{(B)}$ . Assume that (3.6) holds and moreover that  $C_0/A < 1/2$ . Let*

$$\begin{aligned} F(g) &= -W(X) + \left( (1 \otimes \tau + \tau \otimes 1) (Tr(\mathcal{J} \mathcal{D} \Sigma g)) \right. \\ &\quad \left. - \left\{ W(X + \mathcal{D} \Sigma g) - W(X) + Q(\Sigma g) + \frac{1}{2} \mathcal{D} \Sigma g \# \mathcal{D} \Sigma g \right\} \right) \\ &= -W(X + \mathcal{D} \Sigma g) - \frac{1}{2} \mathcal{D} \Sigma g \# \mathcal{D} \Sigma g + (1 \otimes \tau + \tau \otimes 1) Tr \log(1 + \mathcal{J} \mathcal{D} \Sigma g). \end{aligned}$$

(here we abbreviate  $W(X_1, \dots, X_n)$  by  $W(X)$ , etc.)

Then  $F(g)$  is a well-defined function from  $\mathcal{A}_0^{(A)}$  to  $\mathcal{A}^{(A)}$ . Moreover  $g \mapsto F(g)$  is locally Lipschitz on  $\{g : \|g\|_A < A^2/2\}$ :

$$\begin{aligned} \|F(g) - F(f)\|_A &\leq \|f - g\|_A \left\{ \frac{2}{A^2} \left( \frac{1}{\left(1 - \frac{2\|f\|_A}{A^2}\right) \left(1 - \frac{2\|g\|_A}{A^2}\right)} + 1 \right) \right. \\ &\quad \left. + \|\mathcal{J} W\|_{B \otimes_\pi B} + \frac{n}{2} (\|f\|_A + \|g\|_A) \right\} \end{aligned}$$

and bounded:

$$\|F(g)\|_A \leq \|g\|_A \left\{ \frac{2}{A^2} \left( \frac{1}{1 - \frac{2\|g\|_A}{A^2}} + 1 \right) + \|\mathcal{J} W\|_{B \otimes_\pi B} + \frac{n}{2} \|g\|_A \right\} + \|W\|_A$$

In particular, if

$$(3.7) \quad \begin{cases} A > 4, & 0 < \rho \leq 1 \\ \|W\|_A < \frac{\rho}{12n} \\ \|\mathcal{J}W\|_{(A+\rho)\otimes_\pi(A+\rho)} < \frac{1}{8} \end{cases}$$

then  $F$  takes the ball  $\{g : \|g\|_A < \frac{\rho}{4n}\}$  to itself and is uniformly contractive with constant  $\lambda \leq \frac{7}{8}$  on that ball.

*Proof.* Note that  $\|\mathcal{D}\Sigma g\|_A \leq \|g\|_A$  and so  $\|X + \mathcal{D}\Sigma g\| \leq B$ . The Lipschitz property then follows from Lemmas 3.11 and 3.9, as well as the estimates:

$$\begin{aligned} \|W(X + \mathcal{D}\Sigma g) - W(X + \mathcal{D}\Sigma f)\|_A &\leq \|\mathcal{J}W\|_{B\otimes_\pi B} \|\mathcal{D}\Sigma g - \mathcal{D}\Sigma f\|_A \\ &\leq \|\mathcal{J}W\|_{B\otimes_\pi B} \|g - f\|_A \end{aligned}$$

and

$$\begin{aligned} \left\| \frac{1}{2} \mathcal{D}\Sigma g \# \mathcal{D}\Sigma g - \frac{1}{2} \mathcal{D}\Sigma f \# \mathcal{D}\Sigma f \right\|_A &\leq \frac{1}{2} \|\mathcal{D}\Sigma g \# (\mathcal{D}\Sigma g - \mathcal{D}\Sigma f)\|_A \\ &\quad + \frac{1}{2} \|\mathcal{D}\Sigma f \# (\mathcal{D}\Sigma g - \mathcal{D}\Sigma f)\|_A \\ &\leq \frac{n}{2} \|\mathcal{D}\Sigma g - \mathcal{D}\Sigma f\|_A (\|\mathcal{D}\Sigma g\|_A + \|\mathcal{D}\Sigma f\|_A) \\ &\leq \frac{n}{2} \|g - f\|_A (\|g\|_A + \|f\|_A). \end{aligned}$$

The estimate on  $\|F(g)\|_A$  follows from the identity  $F(0) = -W$  and the Lipschitz property.

Assuming that (3.7) holds, so that  $A > 4$  and  $\|g\|_A < \frac{\rho}{4n} < \rho \leq 1$ , we have that

$$\frac{2}{A^2} \left( \frac{1}{\left(1 - \frac{2\|f\|_A}{A^2}\right) \left(1 - \frac{2\|g\|_A}{A^2}\right)} + 1 \right) < \frac{1}{8} \left( \frac{64}{49} + 1 \right) < \frac{1}{2}$$

and so the Lipschitz constant of  $F$  is bounded by  $\frac{1}{2} + n\frac{1}{4n} + \|\mathcal{J}W\|_{(A+\rho)\otimes_\pi(A+\rho)} < \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ . Also,

$$\|F(g)\| < \frac{\rho}{4n} \left\{ \frac{1}{8} \left( \frac{8}{7} + 1 \right) + \frac{1}{8} + \frac{1}{8} \right\} + \|W\|_A < \left( \frac{2}{3} + \frac{1}{3} \right) \frac{\rho}{4n} = \frac{\rho}{4n}$$

so that the image of the ball  $\{g : \|g\|_A < \frac{\rho}{4n}\}$  is contained in that ball.  $\square$

**3.6. Existence of  $g$ .** We will now consider the case that  $\tau$  is the semicircle law. Then (3.6) holds with  $C_0 = 2$ . We remind the reader that  $\Pi, \Sigma$ , are defined in section 2.2 whereas  $\mathcal{S}$  is defined in Definition 2.2.

**Proposition 3.13.** *Let  $\tau$  be the semicircle law. Assume that for some  $A$ ,  $W \in \mathcal{A}_0^{(A)}$  is cyclically symmetric and that conditions (3.7) are satisfied, i.e.,*

$$\begin{cases} A > 4, & 0 < \rho \leq 1 \\ \|W\|_A < \frac{\rho}{12n} \\ \|\mathcal{J}W\|_{(A+\rho)\otimes_\pi(A+\rho)} < \frac{1}{8} \end{cases}$$

Then there exists  $\hat{g}$  and  $g = \Sigma \hat{g}$  (so that  $\hat{g} = \mathcal{N}g$ ) with the following properties:

- (i) Both  $\hat{g}$  and  $g$  belong to the completion of  $\mathcal{A}_0$  with respect to the norm  $\|\cdot\|_A$ ;
- (ii)  $\hat{g}$  satisfies the equation  $\hat{g} = \mathcal{S}\Pi F(\hat{g})$
- (iii)  $\hat{g}$  and  $g$  depend analytically on  $W$ , in the following sense: if the maps  $\beta \mapsto W_\beta$  are



analytic, then also the maps  $\beta \mapsto \hat{g}(\beta)$  and  $\beta \mapsto g$  are analytic, and  $g \rightarrow 0$  if  $\|W\|_A \rightarrow 0$   
(iv)  $g$  satisfies the equation

$$(3.8) \quad \mathcal{S}\Pi\mathcal{N}g = -W(X) + \mathcal{S}\Pi \left( (1 \otimes \tau + \tau \otimes 1)(\text{Tr}(\mathcal{J}\mathcal{D}g)) - \left\{ W(X + \mathcal{D}g) - W(X) + Q(g) + \frac{1}{2}\mathcal{D}g\#\mathcal{D}g \right\} \right) \\ = \mathcal{S}\Pi \left[ -W(X + \mathcal{D}g) - \frac{1}{2}\mathcal{D}g\#\mathcal{D}g + (1 \otimes \tau + \tau \otimes 1)\text{Tr} \log(1 + \mathcal{J}\mathcal{D}g) \right].$$

or, equivalently,

$$(3.9) \quad -\mathcal{S}\Pi\mathcal{N}g + \mathcal{S}\Pi(1 \otimes \tau + \tau \otimes 1)(\text{Tr}(\mathcal{J}\mathcal{D}g)) = \mathcal{S}\Pi W(X) \\ + \mathcal{S}\Pi \left\{ W(X + \mathcal{D}g) - W(X) + Q(g) + \frac{1}{2}\mathcal{D}g\#\mathcal{D}g \right\}.$$

*Proof.* Observe that Equation (3.8) is equivalent to

$$\mathcal{S}\Pi\mathcal{N}g = \mathcal{S}\Pi F(\mathcal{N}g)$$

with  $F$  the function defined in Corollary 3.12, where we used that  $W$  is cyclically symmetric and in  $\mathcal{A}_0^{(A)}$ . Existence of  $g$  follows, essentially, from the Implicit function Theorem 2.3. We prefer to repeat its proof here for completeness. We set  $\hat{g}_0 = W(X_1, \dots, X_n)$ , and for each  $k > 0$ ,

$$\hat{g}_k = \mathcal{S}\Pi F(\hat{g}_{k-1}),$$

Since  $\mathcal{S}\Pi$  is a linear contraction, the last part of Corollary 3.12 implies that under our hypothesis,  $\mathcal{S}\Pi \circ F$  is uniformly contractive with constant  $7/8$  on the ball  $B = \{\hat{g} : \|\hat{g}\|_A < \frac{1}{4n}\}$ , and takes this ball to itself. It follows that  $\hat{g}_k \in B$  for all  $k$ , and moreover

$$\|\hat{g}_k - \hat{g}_{k-1}\|_A = \|\mathcal{S}\Pi F(\hat{g}_{k-1}) - \mathcal{S}\Pi F(\hat{g}_{k-2})\| < \frac{7}{8}\|\hat{g}_{k-2} - \hat{g}_{k-3}\|_A,$$

so that  $\hat{g}_k$ 's converge in  $\|\cdot\|_A$  to a fixed point  $\hat{g}$ . Since  $\hat{g} \in \mathcal{A}_0^{(A)}$ , so does  $g = \Sigma\hat{g}$ . This proves (i) and (ii).

Since (assuming that  $\beta \mapsto W_\beta$  is analytic) each iterate  $\hat{g}_k$  is clearly analytic in  $\beta$  and the convergence of the Banach-space valued function  $\hat{g}_k(\beta) \in \mathcal{A}_0^{(A)}$  is uniform on any compact disk inside  $|\beta| < \beta_0$ , it follows from the Cauchy integral formula that the limit is also an analytic function. It follows from the norm estimates in Corollary 3.12 (taking  $\rho \rightarrow 0$ ) that the bound on  $\|\hat{g}_k\|_A$  in the iterative procedure goes to zero as  $\|W\|_A \rightarrow 0$ , so that  $\|g\|_A \rightarrow 0$  as  $\|W\|_A \rightarrow 0$ .

Part (iv) follows from the definition of  $F$  and  $\Sigma$ . □

We leave the following to the reader (note that  $\hat{g}$  is clearly cyclically symmetric, and it is not hard to see that this implies that also  $g$  is cyclically symmetric):

**Proposition 3.14.** *Assume that  $W = W^*$  and that  $W$  satisfies the hypothesis of Proposition 3.13. Let  $g$  be the solution to (3.9) constructed in Proposition 3.13. Then  $g$  belongs to the closure of cyclically symmetric polynomials and also satisfies  $g = g^*$ .*

**3.7. The map  $f = \mathcal{D}g$  satisfies Equation (3.4).** From now on, we will assume that  $\tau$  is the semicircle law.

**Theorem 3.15.** *Let  $A > A' > 4$ . Then there exists a constant  $C(A, A') > 0$  depending only on  $A, A'$  and  $n$ , so that whenever  $W = W^* \in \mathcal{A}^{(A+1)}$  satisfies  $\|W\|_{A+1} < C(A, A')$ , there exists an  $f \in (\mathcal{A}^{(A')})^n$  which satisfies (3.4). In addition,  $f = \mathcal{D}g$  for some  $g \in \mathcal{A}^{(A')}$ . The solution  $f = f_W$  satisfies  $\|f_W\|_{A'} \rightarrow 0$  as  $\|W\|_{A+1} \rightarrow 0$ . Moreover, if  $W_\beta$  is a family which is analytic in  $\beta$  then also the solutions  $f_{W_\beta}$  are analytic in  $\beta$ .*

*Proof.* We first note that, by the Cauchy integral formula, if we choose  $A_1$  so that  $4 < A_1 < A$ , then

$$\|\mathcal{D}g\|_{A_1} \leq \frac{n(A+1)}{|1+A-A_1|^2} \|g\|_{A+1}.$$

Similarly, we see that

$$(3.10) \quad \|\mathcal{J}f\|_{A_1 \otimes_\pi A_1} \leq \frac{n(A+1)}{|1+A-A_1|^3} \|g\|_A$$

and also

$$\|\mathcal{J}W\|_{(A_1+1) \otimes_\pi (A_1+1)} \leq \frac{(A+1)}{|A-A_1|^2} \|W\|_{A+1}.$$

Thus we can choose  $C(A, A') > 0$  so that if  $\|W\|_{A+1} < C(A, A')$ , then the hypothesis of Proposition 3.13 is satisfied, and there exists some  $g$  which satisfies

$$\begin{aligned} & -\mathcal{S}\Pi\mathcal{N}g + \mathcal{S}\Pi(1 \otimes \tau + \tau \otimes 1)(\text{Tr}(\mathcal{J}\mathcal{D}g)) \\ & = \mathcal{S}\Pi W(X) + \mathcal{S}\Pi \left\{ W(X + \mathcal{D}g) - W(X) + Q(g) + \frac{1}{2} \mathcal{D}g \# \mathcal{D}g \right\}. \end{aligned}$$

Furthermore, we may assume (by choosing a perhaps smaller  $C(A, A')$ ) that  $f = \mathcal{D}g$  satisfies  $\|\mathcal{J}f\|_{A' \otimes_\pi A'} < 1$ . Applying  $\mathcal{D}$  to both sides and noting that for any  $h$ ,  $\mathcal{D}\mathcal{S}\Pi h = \mathcal{D}h$ , gives us

$$\mathcal{D}\{(1 \otimes \tau + \tau \otimes 1)(\text{Tr}[\mathcal{J}\mathcal{D}g]) - \mathcal{N}g\} = \mathcal{D}(W(X + f)) + \mathcal{D}Q(g) + \mathcal{J}f \# f,$$

which, according to Lemma 3.7 is equivalent to (3.4). The hypothesis of Lemma 3.7 is satisfied since the map  $T : \xi \mapsto \mathcal{J}f \# \xi$  is strictly contractive on  $(\mathcal{A}^{(A')})^n$  because of the bound on  $\mathcal{J}f$  (and so the map  $\text{id} + T$  has no kernel).  $\square$

Since  $Y_W = X + f_W \rightarrow X$  is bounded by  $A$  as  $W \rightarrow 0$  in  $\|W\|_A$ , it follows that for sufficiently small  $\|W\|_A$ , the law of  $Y$  is the unique solution to the Schwinger-Dyson equation with self-adjoint potential  $V = \frac{1}{2} \sum X_j^2 + W$ , and thus the law of  $Y$  is exactly  $\tau_V$  (by [GMS06, Theorem 2.1] or alternatively since, for  $\|W\|_A$  sufficiently small,  $V$  is then  $(c, M)$  convex and the Schwinger-Dyson equation has a unique solution [GS09]). In particular,  $W^*(Y_1, \dots, Y_n) \cong W^*(\tau_V)$ . We thus obtain a free analog of Brenier's theorem:

**Theorem 3.16.** *Let  $A > A' > 4$ , and let  $X_1, \dots, X_n \in (M, \tau)$  be semicircular variables. Then there exists a universal constant  $C = C(n, A, A') > 0$  so that whenever  $W \in \mathcal{A}^{(A+1)}$  satisfies  $\|W\|_{A+1} < C$ , there is an element  $G \in \mathcal{A}^{(A')}$  so that*

$$(Y_1, \dots, Y_n) = (\mathcal{D}_1 G, \dots, \mathcal{D}_n G) \in \mathcal{A}^{(A')}$$

*has law  $\tau_V$ ,  $V = \frac{1}{2} \sum X_j^2 + W$ .*

*Moreover, the Hessian  $\mathcal{J}\mathcal{D}g$  is a strictly positive element of  $M_{n \times n}(M \bar{\otimes} M^{op})$ .*

In particular, there are trace-preserving injections  $C^*(\tau_V) \subset C_{\text{red}}^*(\mathbb{F}_n)$  and  $W^*(\tau_V) \subset L(\mathbb{F}_n)$ .

If the map  $\beta \mapsto W_\beta$  is analytic, then  $Y_1, \dots, Y_n$  are also analytic in  $\beta$ . Furthermore,  $\|Y_j - X_j\|_{A'}$  vanishes as  $\|W\|_{A+1}$  goes to zero.

*Proof.* We use Theorem 3.15 and set  $G = \frac{1}{2} \sum X_j^2 + g$  and the discussion before the statement of the present theorem. The only thing left to prove is that the Hessian of  $G$  is strictly positive. But the Hessian of  $G$  is given by

$$\mathcal{J} \mathcal{D}G = 1 + \mathcal{J} \mathcal{D}g.$$

In the proof of Theorem 3.15, we have chosen  $C$  in such a way that  $\|\mathcal{J} \mathcal{D}g\|_{A' \otimes_\pi A'} < 1$ , which means that  $\mathcal{J} \mathcal{D}G$  is strictly positive.  $\square$

#### 4. SOME APPLICATIONS.

**4.1. Analyticity of the solution to the Schwinger-Dyson equation.** Under the hypothesis of Theorem 3.16, we deduce that if  $\beta \rightarrow W_\beta$  is an analytic family of potentials and  $V_\beta = \frac{1}{2} \sum X_j^2 + W_\beta$ , then there exists an analytic family of elements  $Y_\beta$  whose law  $\tau_\beta$  satisfies the Schwinger-Dyson equation for  $V_\beta$ . We deduce the following corollary, which was already proved in [GS09], by a different method:

**Corollary 4.1.** *Let  $P \in \mathcal{A}$  be a fixed polynomial. Then  $\tau_\beta(P)$  is analytic in  $\beta$  in a neighborhood of the origin.*

**4.2. Isomorphism results.** Applying the implicit function theorem, we can improve Theorem 3.16 at the expense of possibly choosing a smaller bound on  $W$ . Indeed, we let  $(Y_1, \dots, Y_n) = f = \mathcal{D}g$  be as in Theorem 3.16, we can always assume that  $Y_j(0, \dots, 0) = 0$  by replacing each  $Y_j$  with  $Y_j - Y_j(0, \dots, 0)$ . Thus we may assume that  $f(0) = 0$ .

**Theorem 4.2.** *Let  $A > A' > 4$ , and let  $X_1, \dots, X_n \in (M, \tau)$  be semicircular variables. Then there exists a universal constant  $C = C(n, A, A') > 0$  so that whenever  $W \in \mathcal{A}^{(A+1)}$  satisfies  $\|W\|_{A+1} < C$ , there is an element  $G \in \mathcal{A}^{(A')}$  so that:*

- (1) *If we set  $Y_j = \mathcal{D}_j G$ , then  $Y_1, \dots, Y_n \in \mathcal{A}^{(A')}$  has law  $\tau_V$ , with  $V = \frac{1}{2} \sum X_j^2 + W$ ;*
- (2)  *$X_j = H_j(Y_1, \dots, Y_n)$  for some  $H \in \mathcal{A}^{(A')}$ ;*
- (3) *the Hessian  $\mathcal{J} \mathcal{D}G$  is a strictly positive element of  $M_{n \times n}(M \bar{\otimes} M^{\text{op}})$ .*

In particular, there are trace-preserving isomorphisms

$$C^*(\tau_V) \cong C^*(X_1, \dots, X_n), \quad W^*(\tau_V) \cong L(\mathbb{F}_n).$$

**4.2.1. Isomorphism class of algebras with analytic conjugate variables that are close to the generators.**

**Corollary 4.3.** *Let  $X_1, \dots, X_n \in (M, \tau)$  be generators of a von Neumann algebra  $M$  and assume that the Fisher information  $\Phi^*(X_1, \dots, X_n)$  is finite. Assume moreover that  $\xi_j^* = \xi_j = \partial_j^*(X_j)$  belongs to  $\mathcal{A}^{(A+1)}$  for some  $A > 4$ .*

*Then there exists a universal constant  $C = C(n, A)$  so that if  $\|\xi_j - X_j\|_{A+1} < C$ , then  $M \cong L(\mathbb{F}_n)$  and also  $C^*(X_1, \dots, X_n) \cong C^*(S_1, \dots, S_n)$  where  $S_1, \dots, S_n$  are free semicircular variables.*

*Proof.* Letting  $V = \frac{1}{2} \sum \left( \sum_{j=1}^n X_j \xi_j + \xi_j X_j \right)$ , one gets that  $\xi_j = \mathcal{D}_j V$  (cf. [Voi00]). Thus we can write  $V = \frac{1}{2} \sum X_j^2 + W$ ; then for small enough  $\|\xi_j - X_j\|_{A+1}$ ,  $W = W^*$  satisfies the hypothesis of Theorem 4.2.  $\square$

4.2.2. *Voiculescu's conjecture with polynomial potentials.* The following corollary is a partial answer to a conjecture of Voiculescu [Voi06, p. 240] (the full statement of Voiculescu's conjecture is that the isomorphism should hold for arbitrary polynomials  $W$  and arbitrary  $\beta$ ; however, even uniqueness of a free Gibbs state is unknown in that generality!)

**Corollary 4.4.** *Let  $A > 4$  and let  $W \in \mathcal{A}^{(A)}$  be a self-adjoint power series. Let  $V_\beta = \sum X_j^2 + \beta W$ , and let  $\tau_\beta$  be a trace satisfying the Schwinger-Dyson equation with potential  $V_\beta$ . Then there exists a  $\beta_0 > 0$  so that  $W^*(\tau_\beta) \cong W^*(\tau_0) \cong L(\mathbb{F}_n)$  and  $C^*(\tau_\beta) \cong C^*(\tau_0) = C^*(S_1, \dots, S_n)$  for all  $-\beta_0 < \beta < \beta_0$  (here  $S_1, \dots, S_n$  is a free semicircular system).*

4.2.3.  *$q$ -deformed free group factors.* In [BS91], Bozejko and Speicher have introduced a family of von Neumann algebras  $\Gamma_q(\mathbb{R}^n)$ , which are “ $q$ -deformations” of free group factors. For  $q = 0$ ,  $\Gamma_q(\mathbb{R}^n) \cong L(\mathbb{F}_n)$ , but the question of whether an isomorphism like this holds for  $q \neq 0$  remained open. Despite being much-studied ([BS91, BKS97, Nou04, Śni01, Śni04, Shl04, Ric05, Shl09, KN11, Dab10, Avs11] is an incomplete list of results about these factors), the question of the isomorphism class of these algebras for all values of  $q$  remains elusive. Nonetheless, we are able to settle it for small  $q$  (depending on  $n < \infty$ ):

**Corollary 4.5.** *Let  $\Gamma_q(\mathbb{R}^n)$  be the von Neumann algebra generated by a  $q$ -semicircular system  $S_1^{(q)}, \dots, S_n^{(q)}$ ,  $n \in \{2, 3, \dots\}$ . Then there exists a  $0 < q_0 = q_0(n)$  depending on  $n$ , so that  $\Gamma_q(\mathbb{R}^n) \cong \Gamma_0(\mathbb{R}^n) \cong L(\mathbb{F}_n)$  and  $C^*(S_1^{(q)}, \dots, S_n^{(q)}) = C^*(S_1^{(0)}, \dots, S_n^{(0)})$  for all  $|q| < q_0$ .*

*Proof.* This follows from the fact [Dab10, Theorem 34] that the conjugate variables  $\xi_j = \xi_j^* = \mathcal{D}V$  to  $(S_1^{(q)}, \dots, S_n^{(q)})$  exist and that for some  $A > 5$ ,  $\|\xi_j - X_j\|_A \rightarrow 0$  as  $q \rightarrow 0$ .  $\square$

4.2.4. *Free entropy.* Note that under the assumptions of Theorem 4.2, we end up expressing  $X_j$ 's as a convergent power series in  $n$  semicircular variables  $S_1, \dots, S_n$ . By the change of variables formula of Voiculescu [Voi94], we get:

**Corollary 4.6.** *Let  $V = \frac{1}{2} \sum X_j^2 + W$ , and assume the hypothesis of Theorem 4.2. Let  $\tau_V$  be the free Gibbs state with potential  $V$ . Then*

$$\chi(\tau_V) = \chi(\tau) + \tau \otimes \tau[\mathrm{Tr}(\log(\mathcal{J} \mathcal{D}G))]$$

where  $\tau$  is the semicircle law and  $G$  is as in Theorem 4.2.

4.3. **Monotone transport for random matrix models.** Let  $V = \frac{1}{2} \sum X_j^2 + W$  and  $\tau_V$  be the free Gibbs state with potential  $V$ . Assume that  $W$  satisfies the hypothesis of Theorem 4.2 for some  $A > A' > 4$  and let  $F = (F_1, \dots, F_n) = \mathcal{D}G$  be the map in  $(\mathcal{A}^{(A')})^n$  constructed in that theorem. We let

$$V(X_1, \dots, X_n) = \frac{1}{2} \sum X_j^2 + W(X_1, \dots, X_n).$$

on  $\{\max_i \|X_i\| \leq A\}$  and  $V = +\infty$  otherwise.  $V$  is strictly convex for  $\|W\|_{A+1}$  small enough

Let  $\mu_V^{(N)}$  be the measure on  $(M_{N \times N}^{sa})^n$  given by

$$d\mu_V^{(N)} = \frac{1}{Z_N^V} \exp(-N \mathrm{Tr}(V(A_1, \dots, A_n))) dA_1 \dots dA_n$$

where we set  $dA_1 \dots dA_n = \text{Lebesgue measure}$ . Consider also the Gaussian measure

$$d\mu^{(N)} = \frac{1}{Z_N} \exp \left[ -N \mathrm{Tr} \left( \frac{1}{2} \sum A_j^2 \right) \right] dA_1 \dots dA_n.$$

We denote by  $\chi : \mathbb{R} \rightarrow [-A', A']$  a smooth cutoff function so that  $\chi(x) = x$  for  $x \in [-4, 4]$ ,  $|\chi(x)| \leq A'$  and  $\chi$  has a uniformly bounded derivative. On the space of  $n$ -tuples of  $N \times N$  self-adjoint matrices  $(M_{N \times N}^{sa})^n$  with norm no more than 4, consider the functional calculus map

$$\tilde{F}^{(N)} : (A_1, \dots, A_n) \mapsto (F_j(\chi(A_1), \dots, \chi(A_n)))_{j=1}^n.$$

Finally, let  $F^{(N)} = \nabla \Psi^{(N)}$  be the unique monotone transport map on  $(M_{N \times N}^{sa})^n$  so that

$$F_*^{(N)} \mu^{(N)} = \mu_V^{(N)}.$$

**Theorem 4.7.** *Let  $A > A' > 4$  and assume that  $\|W\|_{A+1} < c$  with  $c$  small enough, at least smaller than the constant of Theorem 4.2.*

(i) *With the above notation,*

$$\lim_{N \rightarrow \infty} \int \frac{1}{N} \text{Tr} \left( \left[ F^{(N)} - \tilde{F}^{(N)} \right]^2 \right) d\mu^{(N)} = 0.$$

*In other words, as  $N \rightarrow \infty$ , the “entrywise” monotone transport  $F^{(N)}$  is well-approximated as  $N \rightarrow \infty$  by the “matricial” functional calculus map  $\tilde{F}^{(N)}$ .*

(ii) *Assume in addition that  $W$  is  $(c, A)$ -convex for some  $c > 0$ . Then  $\mathcal{J}F \leq 1$ . Furthermore,  $\frac{1}{N} \text{Tr} \left( \left[ F^{(N)} - \tilde{F}^{(N)} \right]^2 \right)$  vanishes almost surely as  $N$  goes to infinity.*

*Proof.* Let us put, for a probability measure  $\nu$  on  $\mathbb{R}^M$  with density  $p(x_1, \dots, x_M) dx_1 \dots dx_M$ , and  $U : \mathbb{R}^M \rightarrow \mathbb{R} \cup \{+\infty\}$  a convex function,

$$\begin{aligned} H(\nu) &= \int p(x_1, \dots, x_M) \log p(x_1, \dots, x_M) dx_1 \dots dx_M \\ H_U(\nu) &= \int p(x_1, \dots, x_M) \log p(x_1, \dots, x_M) dx_1 \dots dx_M + \int U d\nu. \end{aligned}$$

If  $h$  is an invertible transformation with positive-definite Jacobian  $\text{Jac } h$ , then

$$\begin{aligned} H(h_*\nu) &= H(\nu) - \int \log \det \text{Jac } h \, d\nu \\ &= H(\nu) - \int \text{Tr} \log \text{Jac } h \, d\nu. \end{aligned}$$

Both entropies  $H_U(\nu)$  and  $H(\nu)$  are convex functions of the density of  $\nu$ . Moreover,  $H_U(h_*\nu)$  and  $H(h_*\nu)$  are convex in  $h$  if  $U$  is convex.

It is well-known that  $H_U(\nu)$  is minimized precisely by the Gibbs measure with potential  $U$ :

$$H_U(\nu) = \inf_{\nu'} H_U(\nu') \iff d\nu(x_1, \dots, x_M) = \frac{1}{Z} \exp(-U(x_1, \dots, x_M)) dx_1 \dots dx_M.$$

Similarly, for  $V \in \mathcal{A}$ , let  $\chi(\tau)$  be Voiculescu’s microstates free entropy, and put  $\chi_V(\tau) = \chi(\tau) - \tau(V)$ .

Then using [Voi02b, GS09, Gui09] one has

$$\chi_V(\tau) = \sup_{\tau'} \chi_V(\tau') \iff \tau \text{ is the free Gibbs law with potential } V.$$

Moreover, if we set  $V^{(N)}(A_1, \dots, A_n) = N \text{Tr}(V(A_1, \dots, A_n))$ , then following [GS09, proof of Theorem 5.1] or [GMS06, Section 3.3] we see that  $\mu_V^{(N)}(\frac{1}{N} \text{Tr}(P))$  converges towards  $\tau_V(P)$

for all polynomials  $P$ , in particular the limit does not depend on the cutoff provided  $\|W\|_{A+1}$  is small enough. In fact, as  $\mu_V^{(N)}$  has a strictly log-concave density, Brascamp-Lieb inequalities allow one to show that the matrices under  $\mu_V^{(N)}$  are bounded by 4 with overwhelming probability. Moreover

$$\begin{aligned}\chi_V(\tau_V) &= \lim_{N \rightarrow \infty} \left( \frac{1}{N^2} \log Z_N^V + \frac{n}{2} \log N \right) \\ &= \lim_{N \rightarrow \infty} \left( \frac{n}{2} \log N - \frac{1}{N^2} H_{V^{(N)}}(\mu_V^{(N)}) \right)\end{aligned}$$

Furthermore, we claim that, if  $\tilde{\mu}_V^N = \tilde{F}_*^{(N)} \mu^{(N)}$ , we have

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} H_{V^{(N)}}(\tilde{\mu}_V^{(N)}) = -\chi_V(\tau_V).$$

To see this, we first note that

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \int V^{(N)} d\tilde{\mu}_V^{(N)} = \tau_V(V).$$

Moreover,

$$\frac{1}{N^2} H(\tilde{\mu}_V^{(N)}) = \frac{1}{N^2} H(\mu^{(N)}) - \mathbb{E}_{\mu^{(N)}} \left[ \frac{1}{N^2} \text{Tr} \log \text{Jac } \tilde{F}^{(N)} \right].$$

On the other hand (see e.g. [Gui09]), because of concentration phenomena, as  $\mu^{(N)}$  has a strictly log-concave density, if we take  $\tau$  to be the semicircle law, then

$$\mathbb{E}_{\mu^{(N)}} \left[ \frac{1}{N^2} \text{Tr} \log \text{Jac } \tilde{F}^{(N)} \right] \rightarrow \tau \otimes \tau(\text{Tr} \log \mathcal{J} F).$$

Thus

$$\frac{1}{N^2} H(\tilde{\mu}_V^{(N)}) \rightarrow -\chi(\tau) - \tau \otimes \tau(\text{Tr} \log \mathcal{J} F) = -\chi(\tau_V)$$

so that using Corollary 4.6,

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} H_{V^{(N)}}(\tilde{\mu}_V^{(N)}) = -\chi_V(\tau_V)$$

as claimed. By convexity of entropy as a function of the transport maps, for  $\varepsilon \in [0, 1]$ ,

$$\begin{aligned}& H_{V^{(N)}} \left( \left[ (1 - \varepsilon) F^{(N)} + \varepsilon \tilde{F}^{(N)} \right]_* \mu^{(N)} \right) - H_{V^{(N)}}(\mu_V^{(N)}) \\ & \leq (1 - \varepsilon) H_{V^{(N)}}(F_*^{(N)} \mu^{(N)}) + \varepsilon H_{V^{(N)}}(\tilde{F}_*^{(N)} \mu^{(N)}) - H_{V^{(N)}}(\mu_V^{(N)}) \\ & = \varepsilon (H_{V^{(N)}}(\tilde{\mu}_V^{(N)}) - H_{V^{(N)}}(\mu_V^{(N)})).\end{aligned}$$

On the other hand, let  $\Delta^{(N)} = F^{(N)} - \tilde{F}^{(N)}$  and set

$$D_N = \partial_\varepsilon \Big|_{\varepsilon=0} H_{V^{(N)}} \left( [F^{(N)} + \varepsilon \Delta^{(N)}]_* \mu^{(N)} \right).$$

Since  $H_{V^{(N)}} \left( \left[ (1 - \varepsilon) F^{(N)} + \varepsilon \tilde{F}^{(N)} \right]_* \mu^{(N)} \right)$  has an absolute minimum at  $\varepsilon = 0$ , we deduce that  $D_N \geq 0$ .

Furthermore, assuming that  $\|W\|_{A+1}$  is small enough so that on the set  $\{\max_{1 \leq i \leq n} \|A_i\|_\infty \leq A\}$  the Hessian  $\text{Hess}(V^{(N)})$  is bounded from below by  $cN$  times the identity operator for some

$c > 0$ , and noting that by definition the image of  $\tilde{F}^{(N)}$  is bounded by  $A$  whereas the image of  $F^{(N)}$  composed with  $\chi$  is also bounded by  $A$ ,

$$\begin{aligned}
(4.1) \quad & H_{V^{(N)}} \left( \left[ (1 - \varepsilon)F^{(N)} + \varepsilon\tilde{F}^{(N)} \right]_* \mu^{(N)} \right) - \varepsilon D_N - H_{V^{(N)}}(\mu_V^{(N)}) \\
&= \mathbb{E}_{\mu^{(N)}} \int_0^\varepsilon t \partial_t^2 H_{V^{(N)}} \left( [F^{(N)} + t\Delta^{(N)}]_* \mu^{(N)} \right) dt \\
&= \mathbb{E}_{\mu^{(N)}} \int_0^\varepsilon t \left\{ \text{Tr} \left[ \left( \frac{\text{Jac}(\Delta^{(N)})}{\text{Jac}(tF^{(N)} + (1-t)\tilde{F}^{(N)})} \right)^2 \right] \right. \\
&\quad \left. + \text{Tr} \left( \text{Hess}(V^{(N)}(F^{(N)} + t\Delta^{(N)}))(\Delta^{(N)})^2 \right) \right\} dt \\
&\geq \frac{\varepsilon^2}{2} c N \mathbb{E}_{\mu^{(N)}} (\text{Tr}((\Delta^{(N)})^2)).
\end{aligned}$$

We thus get (recalling that  $D_N \geq 0$ )

$$\begin{aligned}
\frac{c}{2} N \varepsilon^2 \mathbb{E}_{\mu^{(N)}} \text{Tr}((\Delta^{(N)})^2) &\leq \varepsilon (H_{V^{(N)}}(\tilde{\mu}_V^{(N)}) - H_{V^{(N)}}(\mu_V^{(N)})) - \varepsilon D_N \\
&\leq \varepsilon (H_{V^{(N)}}(\tilde{\mu}_V^{(N)}) - H_{V^{(N)}}(\mu_V^{(N)})).
\end{aligned}$$

Since

$$\alpha_N = \frac{1}{N^2} \left| H_{V^{(N)}}(\tilde{\mu}_V^{(N)}) - H_{V^{(N)}}(\mu_V^{(N)}) \right| \rightarrow 0$$

we can now choose  $\varepsilon = \sqrt{\alpha_N} \rightarrow 0$  to conclude that

$$(4.2) \quad \mathbb{E}_{\mu^{(N)}} \left[ \frac{1}{N} \text{Tr}((\Delta^{(N)})^2) \right] \leq \alpha_N^{1/2}$$

which completes the proof of the first point.

Let us now assume that  $W$  is  $(c, A)$ -convex for some  $c > 0$ . Let  $W^{(N)} = N \text{Tr}(W(A_1, \dots, A_n))$  on matrices satisfying  $\max_j \|A_j\|_\infty \leq A'$  and infinite otherwise. Then  $W^{(N)}$  is convex. It then follows from Caffarelli's results [Caf00, Vil03] that the optimal transport map  $F^{(N)}$  taking the Gaussian measure  $\mu^{(N)}$  to the measure with density  $Z_N^{-1} \exp(-\frac{1}{2} N \sum \text{Tr}(A_j^2) + W^{(N)}(A_1, \dots, A_n))$  has Jacobian uniformly bounded by 1.

The map  $(A_1, \dots, A_n) \rightarrow \frac{1}{N} \text{Tr}((\Delta^{(N)}(A_1, \dots, A_n))^2)$  can be viewed as the composition of the map  $R : (A_1, \dots, A_n) \rightarrow \Delta^{(N)}(A_1, \dots, A_n)$  and the map  $Q : (A_1, \dots, A_n) \rightarrow \frac{1}{N} \sum \text{Tr}(A_j^2)$ . The Jacobian of  $R$  is given by  $\mathcal{J} \mathcal{D}g - \text{Hess} \Psi^{(N)}$ , where  $\Psi^{(N)}$  is such that  $F^{(N)} = \nabla \Psi^{(N)}$ . If  $\max_j \|A_j\|_\infty < A$ , this is bounded (as an operator on Hilbert spaces  $M_{N \times N}^n \rightarrow M_{N \times N}^n$  endowed Hilbert Schmidt norms  $\|A\| = \sum_j \text{Tr}(A_j^* A_j)$ ) because  $g$  is a power series and because of Caffarelli's bound. Hence the map  $R$  is Lipschitz with a uniform Lipschitz constant on the set where  $\max_j \|A_j\|_\infty < A$ . The map  $Q$  is Lipschitz with Lipschitz constant of the form  $C/\sqrt{N}$  (see [Gui09, Lemma 6.2]).

Therefore, by concentration inequalities and (4.2), we deduce

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr}((\Delta^{(N)})^2) = 0 \quad \mu^{(N)}\text{-a.s.}$$

We next come back to (4.1) and observe that  $t \text{Jac}(\tilde{F}^{(N)})$  is bounded above uniformly (as a small smooth perturbation of the identity) as well as  $(1-t) \text{Jac}(\tilde{F}^{(N)})$  which is bounded

above by one by [Caf00], so that we get for  $\varepsilon < 1/2$

$$\mu^{(N)} \left[ \frac{1}{N^2} \text{Tr} \{ (\text{Jac}(\Delta^{(N)}))^2 \} \right] \leq \alpha_N / \varepsilon.$$

Taking once again  $\varepsilon = \sqrt{\alpha_N} \rightarrow 0$ , we obtain that

$$\mu^{(N)} \left[ \frac{1}{N^2} \text{Tr} \left\{ \left( \text{Jac}(F^{(N)}) - \text{Jac}(\tilde{F}^{(N)}) \right)^2 \right\} \right] \rightarrow 0.$$

We now apply [Caf00] to conclude that

$$(4.3) \quad \text{Jac } F^{(N)} \leq 1.$$

Let  $Y_N = \text{Jac}(F^{(N)})$ ,  $\tilde{Y}_N = \text{Jac}(\tilde{F}^{(N)})$  be random variables taking values in the space  $M_{n \times n}(\text{End}(M_{N \times N}^{sa})) \cong M_{n \times n}(M_{N \times N} \otimes M_{N \times N})$  endowed with the normalized trace  $\frac{1}{N^2} \text{Tr} \otimes \text{Tr}$ . Both  $Y_N$  and  $\tilde{Y}_N$  are bounded in operator norm, and consequently define elements  $Y$  and  $\tilde{Y}$  in the ultraproduct von Neumann algebra  $\prod_n^\omega M_{n \times n}(M_{N \times N} \otimes M_{N \times N})$ . Since  $\|Y_N - \tilde{Y}_N\|_2 \rightarrow 0$ ,  $Y = \tilde{Y}$ .

Once again, because of concentration,

$$\mu^{(N)} \left[ \frac{1}{N^2} \text{Tr} \left\{ (\text{Jac}(\tilde{F}^{(N)}))^p \right\} \right] \rightarrow \tau \otimes \tau \text{Tr}((\mathcal{J} F)^p)$$

so that the spectrum of  $\mathcal{J} F$  is the same as that of  $\tilde{Y}$  (and so the same as that of  $Y$ ).

By (4.3), the spectrum of  $Y_N$  lies in the interval  $[0, 1]$ . But this implies that also the spectrum of the limiting operator  $Y$  is contained in the same set. Thus  $\mathcal{J} F \leq 1$ .  $\square$

## 5. OPEN QUESTIONS.

We list some open questions that are raised by our results.

- (1) In the classical case, Brenier's theorem [Bre91, Vil03] asserts much more than the statement of our main theorem: the classical analog of the map  $\mathcal{D}g$  gives *optimal* transport from  $\tau$  to  $\tau_V$  for quadratic Wasserstein distance. It would be nice to understand if the same holds true in the non-commutative case (see [BV01] for the extension of the notion of the Wasserstein distance to non-commutative random variables). Note that the map we construct is optimal in the single-variable case  $n = 1$ .
- (2) Brenier gave a heuristic derivation of his theorem through a very general “polar factorization” theorem. Does a theorem like that hold in the non-commutative case? There is an infinitesimal analog of his decomposition (related to the classical Helmholtz decomposition of vector fields) which has been extensively studied by Voiculescu in [Voi02a]. Can non-commutative monotone transport be also obtained in the same way?
- (3) Does the positivity condition  $\mathcal{J} \mathcal{D}G \in \{F > 0 : F \in M_{n \times n}(\tau \otimes \tau^{\text{op}})\}$  on the “Hessian” of  $G$  translate into any kind of convexity properties of  $G$ ?
- (4) What happens to our map  $F = \mathcal{D}G$  in the case that  $V$  is not strictly convex? It can be seen that in the absence of the bounds on  $W$  the isomorphism (or even the embedding  $C^*(\tau_V) \subset C^*(S_1, \dots, S_n)$ ) fails to exist at least on  $C^*$ -level. This is due to the fact that for certain non-convex polynomials, solutions to the Schwinger-Dyson equations may lead to  $C^*$ -algebras with non-trivial projections. But the free semicircular system generates a projectionless  $C^*$ -algebra. Thus failure of convexity



of  $V = \frac{1}{2} \sum X_j^2 + W$  must be “visible” as a defect of regularity of the transport map  $F$ .

- (5) Is the monotone transport map unique? More precisely, let  $X_1, \dots, X_n$  be a semicircular family, and assume that  $Y = (Y_1, \dots, Y_n)$  and  $Y' = (Y'_1, \dots, Y'_n)$  both belong to the  $L^2$ -closure of  $\{\mathcal{D}g(X_1, \dots, X_n) : \mathcal{J} \mathcal{D}g \geq 0\}$ . If the law of  $Y$  is the same as the law of  $Y'$ , is  $Y = Y'$ ?

We can prove a uniqueness statement for our monotone transport if we assume more on the transport map and the “target”  $n$ -tuple  $Y_1, \dots, Y_n$ :

**Theorem 5.1.** *Let  $X_1, \dots, X_n$  be a semicircular family, and assume that  $Y = (Y_1, \dots, Y_n)$  and  $Y' = (Y'_1, \dots, Y'_n)$  both belong to the  $L^2$ -closure of  $\{\mathcal{D}g : \mathcal{J} \mathcal{D}g \geq 0\}$ . Assume moreover that  $Y$  and  $Y'$  are both invertible non-commutative power series in  $X_1, \dots, X_n$ . If  $Y$  and  $Y'$  have as their laws the same free Gibbs law  $\tau_V$  with  $(c, M)$ -convex potential  $V$ , then  $Y = Y'$ .*

*Proof.* The assumption that  $Y, Y'$  are invertible non-commutative power series allow us to apply Voiculescu’s change of variable formula for free entropy [Voi94]:

$$\begin{aligned}\chi(Y) &= \chi(X) + \tau \otimes \tau(\text{Tr}(\log \mathcal{J}Y)) \\ \chi(Y') &= \chi(X) + \tau \otimes \tau(\text{Tr}(\log \mathcal{J}Y')).\end{aligned}$$

Since  $Y$  and  $Y'$  have the same law, they have the same free entropy (which is finite [GS09]). Thus we conclude that

$$\tau \otimes \tau(\text{Tr}(\log \mathcal{J}Y)) - \tau(V(Y)) = \tau \otimes \tau(\text{Tr}(\log \mathcal{J}Y')) - \tau(V(Y')).$$

Let

$$\psi(Y) = \chi(Y) - \chi(X) - \tau(V(Y)).$$

Then  $\psi$  is maximal iff  $Y$  has the law  $\tau_V$  (see e.g. [Voi02b], we sketch the argument for completeness: if one replaces  $Y$  by  $Y + \varepsilon P(Y)$  for some polynomials  $P_1, \dots, P_n$ , then

$$\psi(Y + \varepsilon P(Y)) = \psi(Y) + \varepsilon \{\tau \otimes \tau(\text{Tr}(\mathcal{J}YP)) - \tau(\mathcal{D}V(Y)P)\} + O(\varepsilon^2),$$

and so any maximizer to  $\Psi$  satisfies the Schwinger-Dyson equation and (because  $V$  is  $c, M$ -convex), has the same law as  $Y$ .)

It follows that

$$\tau \otimes \tau(\text{Tr}(\log \mathcal{J}Y)) - \tau(V(Y)) = \chi(Y) - \tau(V(Y)) = \max_Z \chi(Z) - \tau(V(Z)),$$

so a fortiori

$$\tau \otimes \tau(\text{Tr}(\log \mathcal{J}Y)) - \tau(V(Y)) = \sup_{Y \in \{\mathcal{D}g : \mathcal{J} \mathcal{D}g \geq 0\}} \tau \otimes \tau(\text{Tr}(\log \mathcal{J}Y)) - \tau(V(Y)).$$

But since  $\tau \otimes \tau(\text{Tr}(\log \mathcal{J}Y)) - \tau(V(Y))$  is strictly convex in  $Y$  for  $\mathcal{J}Y$  in the positive cone of  $M_{n \times n}(M \bar{\otimes} M^{op})$  (here  $M = W^*(X_1, \dots, X_n)$ ), it follows that there is at most one  $n$ -tuple  $Y$  in the closure of  $Y \in \{\mathcal{D}g : \mathcal{J} \mathcal{D}g \geq 0\}$  which gives this maximal value. Thus  $Y = Y'$ .  $\square$

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